DP-MS-75-44

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A paper Proposed for Publication in the
Journal of The Franklin Institute

This paper was prepared in connection with work under Contract No. AT(07-2)-1 with the U. S. Energy Research and Development Administration.

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Ductile Fracture of Cracked Steel Plates*

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ABSTRACT: A simple relationship between loading for crack initiation, or onset of ductile tear, and crack length is presented for center-cracked plates of mild steel. Formulation of the nonlinear boundary-value problem is based on incremental theory of plasticity for Prandtl-Reuss materials. Quasi-static solutions corresponding to a series of incremental loading conditions are obtained by the method of finite elements. Tests conducted on plates of two types of mild steel agree with numerical results.

I. Introduction

Griffith's theory of brittle fracture (1,2) first clarified the distinction between brittle failure and plastic flow. The fact that most materials are capable of either type of these failure mechanisms was established (3) by large-scale research efforts on ship steel in the 1940's. Orowan (4) suggested

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*The information contained in this article was developed during the course of work under Contract No. AT(07-2)-1 with the U. S. Energy Research and Development Administration.

generalization of Griffith's theory by taking into account the plastic deformation near the crack tip. Irwin (5) considered crack surfaces as discontinuities in the displacement field and made fracture mechanics one of the branches of applied mathematics. Analytical efforts in the 1960's produced a large collection of elegant solutions to mixed boundary-value problems involving cracks, and as a result, quantities related to analyses of crack tip singularities, such as stress intensity factors (5), crack opening displacements (6), and J-integrals (7) were proposed as fracture criteria. For expediency of analysis, these criteria are based on hypothetical materials and geometries, so the quantities involved require experimental determinations to account for crack-tip nonlinearity of material behavior. Feasibility of these criteria depends upon the scale of nonlinear material behavior and the size of the test specimen.

Concerning ductile fracture, Dugdale (8) reported his testing results on large scale yielding and presented a simple relation between the extent of yielding and the remotely applied loading. Goodier and Field (9) constructed a criterion for crack initiation using ultimate strain at the crack tip as the limiting condition. Quantitative assessment of the criterion and further experimental findings were reported by Rosenfield *et al.* (10). Goodier and Kanninen (11) developed a mechanical model for the description of local nonlinear material behavior.

The development of the method of finite elements (12) and the use of modern computers greatly reduced difficulties in formulation and calculation, so recent research findings in fracture mechanics frequently involve nonlinear materials with complex geometries.

This paper presents numerical results as quasi-static responses of center cracked plates of mild steel, and relates the loading at crack initiation to the geometry of the cracked plate. According to incremental theory of plasticity, the boundary-value problem is formulated in plane-stress for an elastic-plastic material exhibiting strain-hardening effect and is solved by the finite element method. The condition limiting the equivalent stress of any one of the elements below the true ultimate stress of the material in a tensile test is used as the criterion of failure.

II. Formulation

For a homogeneous and isotropic solid, the equations of equilibrium in the absence of body forces and inertial effects are

$$\frac{\partial}{\partial x_i}(d\sigma_{ij}) = 0 \tag{1}$$

where σ_{ij} is the component of stress in the direction of coordinate x_i and on a plane normal to x_j. The equations of compatibility define the components of strain ε_{ij} in terms of those of displacement u_i as

$$d\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial x_j}(du_i) + \frac{\partial}{\partial x_i}(du_j)\right] \tag{2}$$

To consider the constitutive relations of a Prandtl-Reuss material (13), the true stress-true strain curve is simplified by a linearly elastic segment with Young's modulus, E, and a plastic segment with a strain hardening rate H. During loading, the material initially yields at an equivalent or true stress $\bar{\sigma} = Y_o$ and fails at $\bar{\sigma} = \bar{\sigma}_{ult}$. The elastic portions of the strain components, signified by the superscript e, follow the Hookian relations:

$$\varepsilon_{ij}^{e} = \frac{1}{E} \left[(1 + \nu)\, \sigma_{ij} - 3\nu\sigma\delta_{ij} \right] \tag{3}$$

where ν is the Poisson ratio, and $\delta_{ij} = 1$ for $i = j$ or $\delta_{ij} = 0$ for $i \neq j$; the hydrostatic stress component is $\sigma = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$. The plastic portions of the strain components, signified by superscript p, follow the Prandtl-Reuss relations:

$$d\varepsilon_{ij}^{p} = d\varepsilon_{ij} - d\varepsilon_{ij}^{e} = \frac{3}{2} \frac{\sigma'_{ij}}{H} \frac{d\bar{\sigma}}{\bar{\sigma}} \quad \text{for} \quad \begin{array}{l} d\bar{\sigma} \geq 0 \\ \bar{\sigma} \geq Y_r \end{array} \tag{4}$$

where $\sigma'_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ are the reduced stress components. These nonlinear differential equations are valid only during loading, $d\bar{\sigma} \geq o$, and for $\bar{\sigma} > Y_r$; i.e., Mises yield criterion for r-th loading cycle is satisfied,

$$\bar{\sigma} = \left(\frac{3}{2} \sigma'_{ij} \sigma'_{ij} \right)^{\frac{1}{2}} \geq Y_r \tag{5}$$

where summation is required for repeated subscripts.

The response of a rectangular plate of width W with a center crack of length L to loading increments ΔF applied remotely and quasi-statically in a direction normal to the crack is formulated as a mixed boundary-value problem. Figure 1 shows a quarter of the plate, OABCD, where OC = 0.5W, and OD = 0.5L. In Cartesian coordinates (x_1, x_2), the following boundary conditions apply for plane-stress consideration:

$$
\begin{aligned}
& u_1(0, x_2) = 0 \\
& \sigma_{12}(x_1, C) = 0, \qquad \sigma_{22} = (x_1, C) = F \\
& \sigma_{11}(0.5W, x_2) = \sigma_{21}(0.5W, x_2) = 0 \\
& \sigma_{12}(x_1, 0) = \sigma_{22}(x_1, 0) = 0, \qquad 0 \leq 2x_1 \leq L \\
& u_2(x_1, 0) = 0 \qquad L \leq 2x_1 \leq W
\end{aligned}
\tag{6}
$$

III. Method of Solution

The quarter plate is divided into 262 triangular elements connected by 149 nodal points. For plane-stress calculation, it is convenient to label the nodal points by even numbers such that the horizontal and vertical components of force or displacement are indicated by consecutive odd and even subscripts, respectively. For a small element defined by nodes i, j, and k, the compatibility relations (2) are approximated by

$$[\Delta\varepsilon] = \frac{1}{2a}[B][\Delta U] \tag{7}$$

where a is the area of triangle i j k, the column matrices $[\Delta\varepsilon]$ and $[\Delta U]$ contain $(\Delta\varepsilon_{11}, \Delta\varepsilon_{22}, \Delta\varepsilon_{12})$ and $(\Delta U_{i-1}, \Delta U_i, \Delta U_{j-1}, \Delta U_j,$

ΔU_{k-1}, ΔU_k), respectively, and

$$[B] = \begin{bmatrix} x_j - x_k & 0 & x_k - x_i & 0 & x_i - x_j & 0 \\ 0 & x_{k-1} - x_{j-1} & 0 & x_{i-1} - x_{k-1} & 0 & x_{j-1} - x_{i-1} \\ x_{k-1} - x_{j-1} & x_j - x_k & x_{i-1} - x_{k-1} & x_k - x_i & x_{j-1} - x_{i-1} & x_i - x_j \end{bmatrix}$$

Similarly, the stress components can be approximated by small differences of nodal forces, ΔF

$$[\Delta\sigma] = \frac{2}{h}[B][\Delta F] \tag{8}$$

where h is the plate thickness. The matrix form of the constitutive relations can be written as

$$[\Delta\sigma] = [D][\Delta\varepsilon] \tag{9}$$

where, corresponding to the elastic relations (3)

$$[D] = [D]^e = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1+\nu) \end{bmatrix} \tag{10}$$

or corresponding to the plastic relations (4)

$$[D] = [D]^p = \frac{E}{Q}\begin{bmatrix} 2P + \sigma_{22}'^2 & 2\nu P - \sigma_{11}'\sigma_{22}' & -\frac{\sigma_{11}' + \nu\sigma_{22}'}{1+\nu}\sigma_{12} \\ 2\nu P - \sigma_{11}'\sigma_{22}' & 2P + \sigma_{11}'^2 & -\frac{\sigma_{22}' + \nu\sigma_{11}'}{1+\nu}\sigma_{12} \\ -\frac{\sigma_{11} + \nu\sigma_{22}}{1+\nu}\sigma_{12} & -\frac{\sigma_{22}' + \nu\sigma_{11}'}{1+\nu}\sigma_{12} & \frac{Q + 2(1-\nu)\sigma_{12}}{2(1+\nu)} \end{bmatrix} \tag{11}$$

$$\text{where } P = \frac{2H}{9E}\bar{\sigma}^2 + \frac{1}{1+\nu}\sigma_{12}^{2},$$

$$Q = R + 2(1-\nu^2)P,$$

$$R = \sigma_{11}^{'2} + 2\nu\sigma_{11}^{'}\sigma_{22}^{'} + \sigma_{22}^{'2}\text{, and}$$

$$\bar{\sigma}^2 = \sigma_{11}^{2} - \sigma_{11}\sigma_{22} + \sigma_{22}^{2} + 3\sigma_{12}^{2}. \tag{12}$$

$[D]^P$ is valid only if $(2\sigma_{11} - \sigma_{22})\Delta\sigma_{11} + (2\sigma_{22} - \sigma_{11})\Delta\sigma_{22} + 6\sigma_{12}\Delta\sigma_{12} > 0$ and $\bar{\sigma} \geq Y_r$.

When Eqs. (7), (8), and (9) are combined, the nodal forces external to the element in consideration can be calculated from given displacements by

$$[\Delta F] = [g][\Delta U] \tag{13}$$

where $[g] = \frac{h}{4a}[B]^T[D][B]$ and T indicates transpose of the matrix. Superposition of 262 matrix equations similar to Eq. (13) leads to

$$[\Delta F] = [G][\Delta U] \tag{14}$$

where $[\Delta F]$ and $[\Delta U]$ contain 298 resulting forces and displacements of the 149 nodal points, and [G] is a 298 x 298 square matrix dependent on [g]. Applying the boundary and equilibrium conditions, Eqs. (6), (1), and (14) can be solved for 298 unknown quantities in $[\Delta F]$ and $[\Delta U]$. By Eqs. (7), (8), and (9), the states of stress and strain for each element are, in turn, obtained.

For elastic response, Eq. (14) is solved for any given load F, and the solution is linearly scaled by setting $\sigma = Y_o$ for the element whose equivalent stress is the greatest. Due to nonlinearity of $[D]^P$, further loading is applied in terms of small increments ΔF. Instead of using the method of controlled yielding of elements one by one and determining each load increment prior to solving Eq. (14) as proposed by Yamada, Yoshimura, and Sakurai (14), the magnitudes of load increments are assigned arbitrarily, so yielding of more than one element at a time is allowed for expediency. States of stress obtained from the previous step are stored, and together with the current solution in terms of change of the stress states, the correct constitutive matrix of Eq. (9) can be chosen according to the yield and loading criteria before finding the solution for the next load increment. The number of elements that change properties during each loading step is kept small by choosing a sufficiently small load increment. Whenever one of the elements reaches the true ultimate stress, the corresponding load is regarded as that of crack initiation, since there exists no solution for higher loading without modifying the mixed conditions arising from the newly created traction-free boundary.

IV. Results

Numerical solutions were obtained for six different ratios of the width of the plate to the length of the center crack, W/L, for two types of mild steels, A285 Grade B and ASTM 516 Grade 70, whose true stress-true strain curves are simplified by line segments as shown in Fig. 2. All grid patterns for the finite element calculation are similar; the size of elements changes gradually from the tip of the crack to the remote area. A typical pattern is shown in Fig. 1. Figure 3 gives the quasi-static solutions of six A285 Grade B steel plates with different lengths of crack. The curves surrounding the crack tips indicate contours of elastic-plastic transition at increasing levels of applied load labeled in units of stress. Similar results for ASTM 516 Grade 70 steel plates are shown in Fig. 4.

Plotting the maximum applied load corresponding to failure stress for one of the elements, σ_f, against L/W, the numerical results are found to fit a simple parabolic relation, as shown in Fig. 5,

$$\sigma_f = Y_o(1-L/W)^{\frac{1}{2}}$$

depending on only one parameter, the initial yield stress, for material characterization. Validity of this relationship is supported by tests that were made on three kinds of steels at room temperature at Savannah River Laboratory (SRL). Gensamer (15) derived a parabolic relation similar to Eq. (15) based on energy considerations.

Acknowledgement

The experimental work of Dr. H. B. Peacock of SRL in support of this work is gratefully acknowledged by the author.

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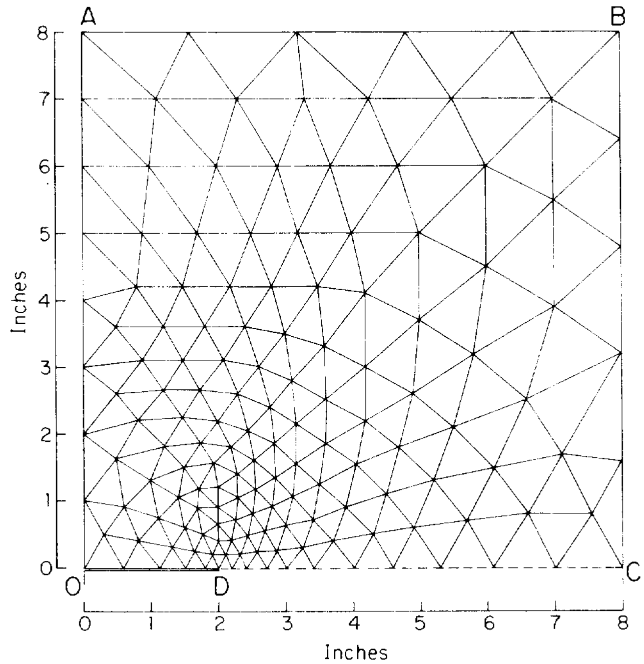

Fig. 1. Grid pattern for 262 elements and 149 nodes.

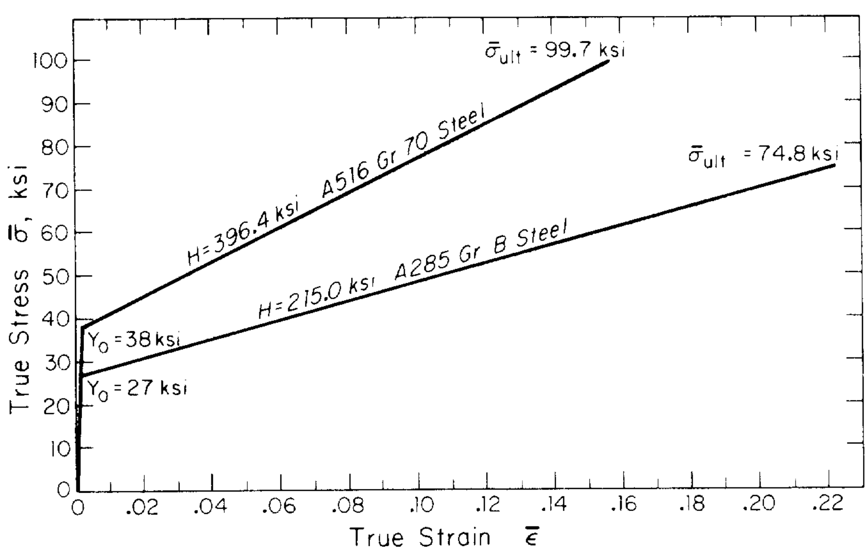

Fig. 2. True stress-true strain relations.

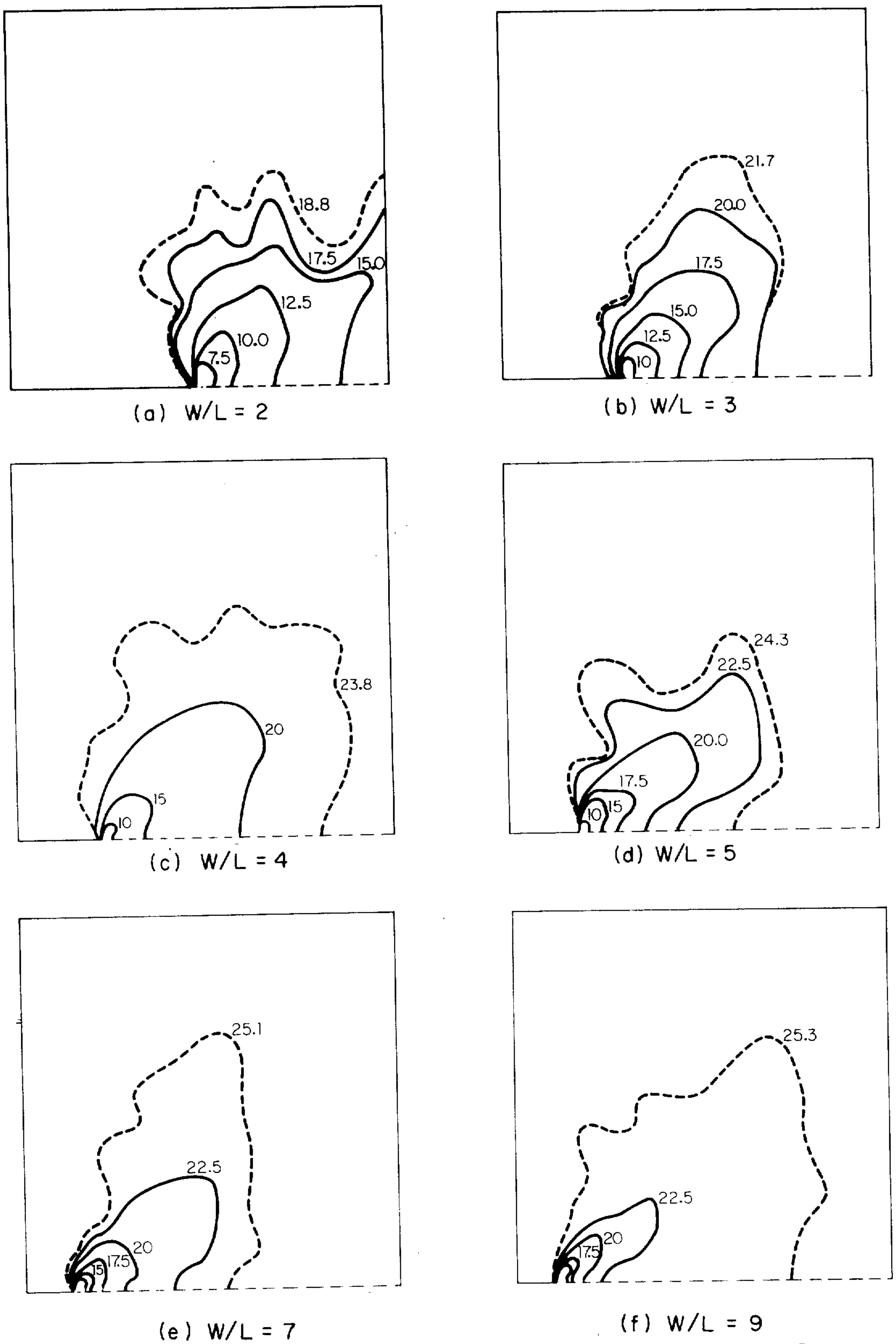

Fig. 3. Elastic-plastic responses of A285 Grade B steel plates. Numbers indicate applied stresses in units of ksi.

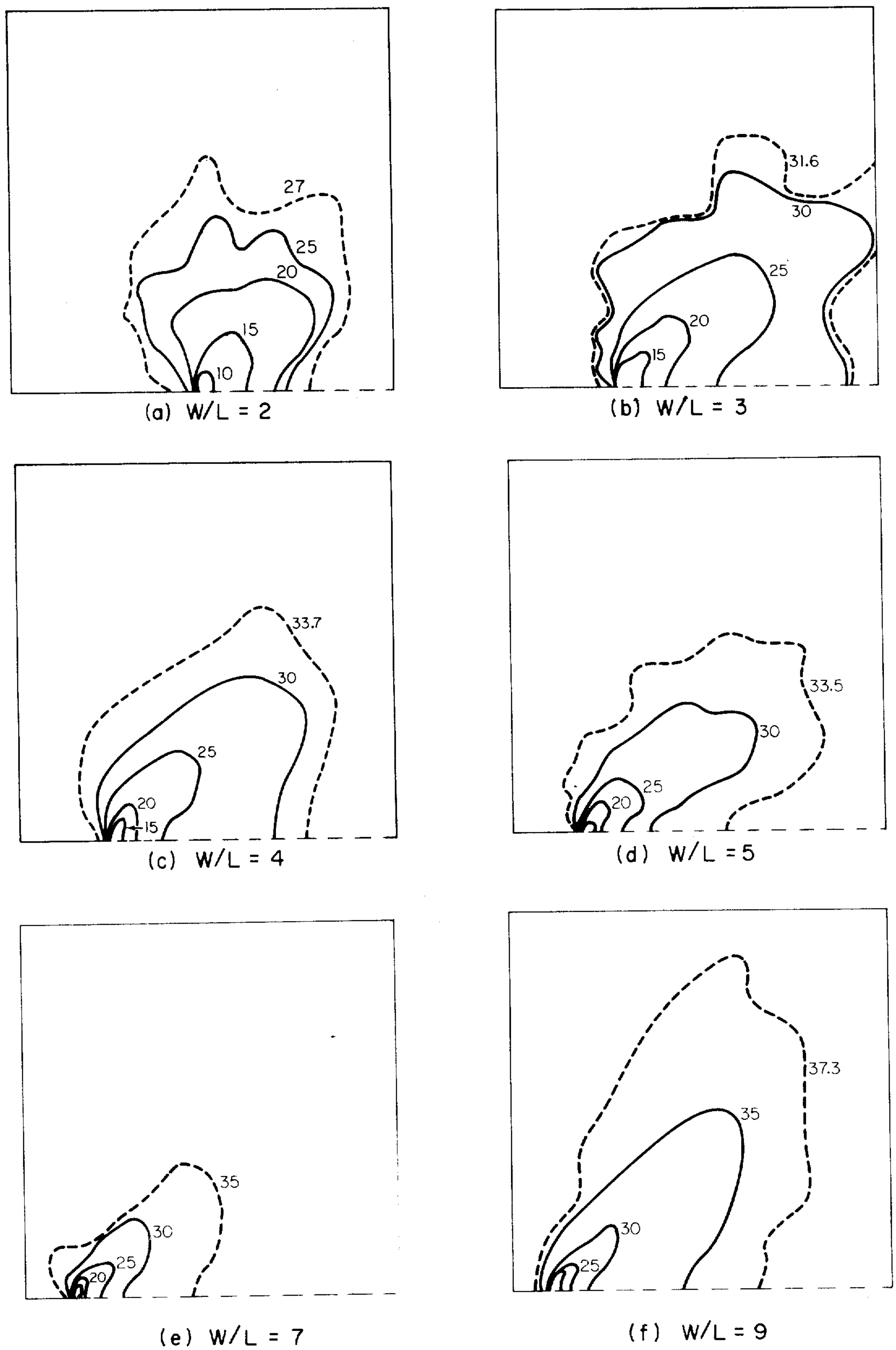

Fig. 4. Elastic-plastic responses of ASTM 516 Grade 70 steel plates. Numbers indicate applied stresses in units of ksi.

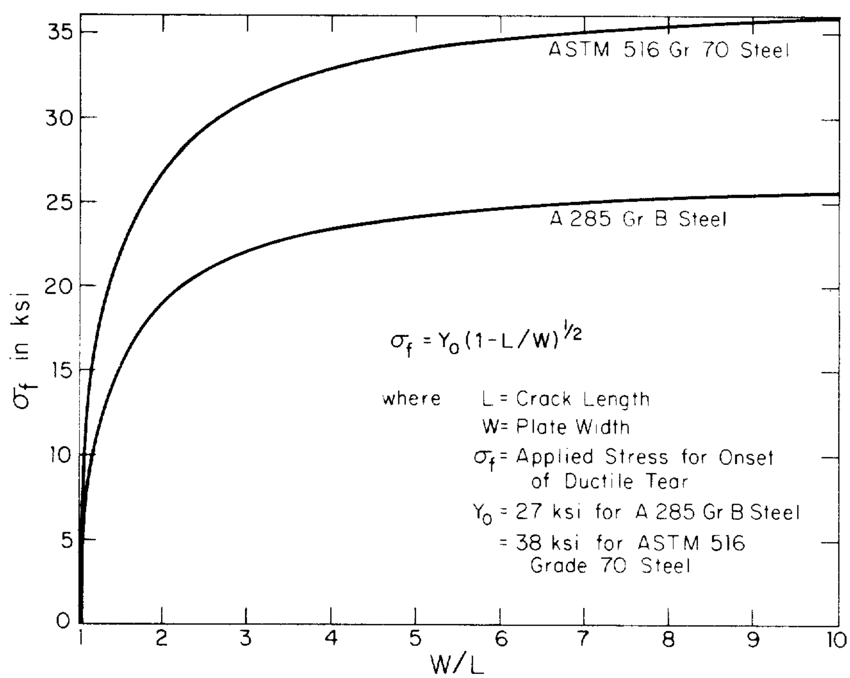

Fig. 5. Ductile initiation stress vs. crack geometry.

LIST OF FIGURES