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ANALYSIS OF HYDRAULIC CONDUCTIVITY CALCULATIONS

R. E. Green and J. C. Corey

Savannah River Laboratory
E. I. du Pont de Nemours and Co.
Aiken, South Carolina 29801

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SOIL BRIEF

A revised equation based on the pore-interaction model of Marshall was shown to accurately predict hydraulic conductivity for glass beads and a loam soil from the pressure-water content relationships of these porous materials.

ABSTRACT

Equations by Marshall and by Millington and Quirk for calculating hydraulic conductivity from pore-size distribution data are dependent on an arbitrary choice of the exponent on the porosity term and a correct estimate of residual water. A modified equation avoids these arbitrary choices and satisfactorily predicts the relationship between hydraulic conductivity and water content for glass beads and soil.

ANALYSIS OF HYDRAULIC CONDUCTIVITY CALCULATIONS[1]

R. E. Green and J. C. Corey[2]

In recent years several investigators have explored the possibility of predicting the hydraulic conductivity of porous materials from pore-size distribution data. Interest in such predictions is warranted inasmuch as the conductivity-water content relationship, $K(\theta)$, is relatively difficult to measure while the pore-size distribution is characterized easily by the standard measurement of water content versus pressure. Childs and Collis-George (1950), Marshall (1958), and Millington and Quirk (1959, 1960, 1961) developed equations for this purpose, and Nielsen et al. (1960), Jackson et al. (1965), Sharma (1966), and Kunze et al. (1968) have tested the equations against experimental data. Some of these tests have indicated the superiority of the Millington and Quirk method when a matching factor is used. Widespread use of conductivity calculations has been discouraged by conflicting published results on the optimum value of the exponent on the porosity term.

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[2] Associate Professor of Soil Science, University of Hawaii, and Research Physicist, Savannah River Laboratory. The work was done while the senior author was an Oak Ridge Associated Universities Research Participant at the Savannah River Laboratory.

This paper (a) evaluates the Marshall and Millington and Quirk equations and (b) proposes an improved method of calculation.

Equation (1) is a generalized form of the Marshall and the Millington and Quirk equations modified to give the $K(\theta)$ relationship when the calculated saturated conductivity is matched with an experimental value. This form of the equation is similar to that given by Kunze et al. (1968).

$$K(\theta)_i = \frac{K_s}{K_{sc}} \cdot \frac{30\,\gamma^2}{\rho g\,\eta} \cdot \frac{\varepsilon^p}{n^2} \sum_{j=i}^{m} \left[(2j + 1 - 2i)\, h_j^{-2} \right] \qquad i = 1, 2, \ldots, m \tag{1}$$

where:

$K(\theta)_i$	is the calculated conductivity for a specified water content or pressure (cm/min),
θ	is the water content (cm^3/cm^3),
i	denotes the last water content class on the wet end, e.g., $i = 1$ identifies the pore class corresponding to the saturated water content, and $i = m$ identifies the pore class corresponding to the lowest water content for which conductivity is calculated,
K_s/K_{sc}	is the matching factor (measured saturated conductivity/ calculated saturated conductivity),
γ	is the surface tension of water (dynes/cm),
ρ	is the density of water (g/cm^3),
g	is the gravitational constant (cm/sec^2),
η	is the viscosity of water (g/cm sec),
ε	is the porosity (cm^3/cm^3), defined in various ways depending on the method of calculation,

p is a parameter that accounts for interaction of pore classes ($1 \le p \le 2$),

n $\geqq m$ (see text); n may or may not vary with θ depending on the calculation method, and

h_j is the pressure for a given class of water-filled pores (cm of H_2O).

Variations in Calculation Methods

The manner in which ϵ and n are varied or held constant leads to three methods (I, II, and III) of calculating $K(\theta)$. All three methods involve the use of a matching factor, as implied by equation (1). Although Marshall and Millington and Quirk did not use matching factors, later workers (Jackson et al., Kunze et al.) have modified the Millington and Quirk calculation by using a matching factor. The three methods are summarized in Table 1.

In Method I, ϵ is the water-filled porosity at each water content and n is the actual number of pore classes used to calculate $K(\theta)$ at a given water content (e.g., if the total number of pore classes at saturation ($\theta_s = 0.50$ cm^3/cm^3) is 20, then $n = 20$ and $\epsilon = 0.50$ cm^3/cm^3; at 50% of saturation $n = 10$ and $\epsilon = 0.25$ cm^3/cm^3). This variation in n with θ is a logical result of Marshall's derivation.

In Method II, ϵ is the water-filled porosity at each water content and n is the total number of pore classes at $\epsilon = \theta_s$ (i.e., n is a constant even though the cross-sectional area available for flow is decreasing with unsaturation). This method was developed by Millington and Quirk to describe a reduction in conductivity at low water contents in excess of that predicted by Marshall's equation. In their 1961 paper, they state that "for the computation of unsaturated permeability the value of ϵ taken is that of the liquid-filled pore space and the r^2 series is commenced at the appropriate pore class, that is the largest pore containing liquid." They do not state explicitly that the term m^{-2} in their equation (1) (which corresponds to n^{-2} in equation (1) of this paper) remains constant as the value of ϵ is reduced with unsaturation, but the constancy of m is inferred in their equations for relative permeability of materials with very narrow pore-size distributions. In their earlier paper (1959), Millington and Quirk explain their rationale for allowing ϵ to vary with n constant: "The basis for the variation of ϵ is due to changes in the water velocity which are not accounted for in the r^2 term...".

In Method III, proposed here, ϵ is the water-saturated porosity (i.e., $\epsilon = \theta_s$) and n is the total number of pore classes at $\epsilon = \theta_s$. Our definitions for ϵ and n arise from Marshall's derivation. Marshall's model of pore interaction results in a series of r_i^2 terms each of which has the coefficient $(\epsilon/n^2)(2i - 1)$, where $i = 1, 2, \ldots, n$. At each level of unsaturation in Method I, ϵ takes on a new value ($\epsilon = \theta$) and n is adjusted to correspond to the number of water-filled pore classes. On the other hand in Method III, ϵ/n^2 is constant; ϵ refers to the total porosity and n to the total number of interacting pore classes, regardless of the degree of unsaturation. The two approaches are consistent in concept, but lead to different results in some cases. The ϵ^2/n^2 term in Marshall's equation is the product of ϵ in Poiseuille's equation (when applied to porous media) and the ϵ/n^2 which results from the pore-interaction model.

In equation (1), we have used ϵ^p/n^2 instead of ϵ^2/n^2 to make the equation applicable to all three methods because the exponent may have values other than $p = 2$. The exponent p in ϵ^p has been assigned values of 2 (Marshall), 1.33 (Millington and Quirk), and 1 (Kunze et al.). The derivations of both Marshall and Millington and Quirk could lead to $1 \leq p \leq 2$, depending on the nature of pore interaction described by the physical model. Because both derivations are based largely on Poiseuille's Law, which has recognized limitations in porous media (Philip, 1958), the value chosen for p appears arbitrary and is best determined by the weight of experimental evidence.

Correction for Residual Water

In the derivations of both Marshall and Millington and Quirk, ϵ refers to the water-filled pores at any given θ. $\epsilon = \theta$ is a satisfactory estimate of the water-filled porosity if all of the water designated by θ is sufficiently mobile to contribute to the conductivity at a given water content. This estimate of water-filled porosity may introduce serious errors in calculations for fine-textured soils having 20 to 30 percent of relatively immobile water at low pressure (e.g., -1 bar). In subsequent discussions, this relatively immobile water is called residual water and designated θ_R. Sharma (1966) found that calculations with Millington and Quirk's method were improved when the residual water correction was made.

The correction for residual water can be accomplished in two different ways:

A. Usually the input experimental data, which specify the pressure - water content relationship for a given soil, cover only that part of the total θ-range considered significant to liquid water movement. In this case, ϵ in equation (1) would be given by $\theta - \theta_R$ for Methods I and II and by $\theta_s - \theta_R$ for Method III, as shown in Table 2. Then n equals m in equation (1), and refers to the number of pore classes associated with the partial θ range. Millington and Quirk (1961) have applied such a correction in their calculations for a silica plug.

B. If the <u>total</u> water content range is considered in the calculation, then n equals the total number of classes between $\theta = 0.0$ and $\theta = \theta_s$. In this case, the correction for residual water can be made by eliminating, in the summation series of equation (1), pressure terms associated with the residual water content range. Thus, $n \neq m$ in equation (1) because m gives the total number of terms in the series at saturation and n is the number of classes assigned to the <u>total</u> water content range (i.e., $n > m$). Marshall (1958) used this approach in his sample calculation in which $n = 14$ but only 12 terms were summed in the series used to calculate the hydraulic conductivity at saturation. In essence, Marshall's calculation implicitly corrected for a residual water content of about 0.075 cm^3/cm^3 (determined from his Figure 1), although he did not explicitly discuss the need for such a correction.

Correction for residual water is important in view of the likelihood of error in computing conductivity when the input data cover only <u>part</u> of the total θ range. Failure to correct for θ_R

will result in $K(\theta)$ values which are much too high at low water contents when calculated by Method I, as shown in the following example:

> If the water-content range considered in a calculation was 0.20 to 0.50 cm^3/cm^3 and this θ-range was divided into 30 equal increments, each increment would have a value of 0.01 cm^3/cm^3. In the calculation of $K(\theta)$ at the water content corresponding to the last increment (dry end), the term ϵ^p/n^2 would have the value $(0.21)^p/1$, while in reality the appropriate term should be closer to $(0.01)^p/1$. Adjusting ϵ by letting $\epsilon = \theta - \theta_R$ would give the correct value. Approach B, in this example, would have 50 increments of 0.01 cm^3/cm^3, and if the calculations were done only for water contents between 0.20 and 0.50, ϵ^p/n^2 would have a value of $(0.21)^p/(21)^2$ for the lowest water content class. The $\underset{p \to 2}{\text{limit}}\ (0.21)^p/(21)^2 = (0.01)^2$, the same value obtained with correction A in which $\epsilon = \theta - \theta_R$.

Method II calculations for a partial θ range should not be as seriously affected by failure to correct for residual water as Method I, because n is constant in Method II.

Method III calculations are unaffected by corrections for θ_R because ϵ^p/n^2 is constant for all water contents. Thus, correction for residual water in Method III will affect only the value of the matching factor (K_s/K_{sc}).

Comparisons With Experimental Data

We calculated hydraulic conductivities with equation (1) by the three methods and compared the results with experimental data of Topp and Miller (1966) and Elrick and Bowman (1964). The data for monodispersed glass beads by the former authors were used in tests by Kunze, et al. (1968). Glass beads provide an extreme case for a porous media because the pore size distribution is unusually narrow (nearly all the water was removed between pressures of 40 and 50 cm of water). Elrick and Bowman's data for Guelph loam were used in a previous test of various conductivity equations by Jackson, et al. (1965). These two sets of data were chosen (a) because of their availability in the literature and (b) to provide continuity between this paper and previous analyses of conductivity calculations.

The computer program developed to do the calculations was based on one provided by Dr. R. J. Kunze. The changes made in his program significantly alter the calculations. Variations of our program gave computations by the three methods with the two types of correction for residual water. The effect of changes in n (5, 10, 20, and 50) and p (1 and 2) were also evaluated. Although the exponent p = 1.33 has been used extensively in previous studies, we did not include it because results for 1.33 would be between values for p = 1 and p = 2.

Calculations by the three methods with n values between 5 and 50 showed that the choice of n was not critical. The largest effect noted was a 1.4-fold difference in $K(\theta)$ obtained at an intermediate water content on Guelph loam. Inconsistencies were evident in the direction of the effect of n on conductivities at low water contents; this was apparently due to the interaction between the effect of n on the assignment of pressure classes and the influence of n on the behavior of ϵ^p/n. A value of n = 20 was chosen to calculate the results shown in Figures 1 and 2. The principal consideration in choosing n for the calculations is accurate representation of the actual water content-pressure curve, which determines the pressure values assigned to each pore class. Calculation inaccuracies would be expected for $n < 10$.

Our results for Topp and Miller's data are shown in Figure 1 and for Elrick and Bowman's data in Figure 2. The data points in Figure 1 were read from the uppermost conductivity curve corresponding to the desorption "main branch" in Topp and Miller's Figure 2. The calculated curves shown for Methods I and II are those obtained with the value of p giving the best results (Method I, p = 2; Method II, p = 1). Method III gave the same results for both values of p.

Comparison of calculated with experimental data in Figures 1 and 2 demonstrates the differences between the three methods of calculation.

1. Method I gave satisfactory results with p = 2. It requires the correction for θ_R by either technique A or B (defined in Table 2); both techniques gave the same results. The conductivities calculated with p = 1 (not shown) were much too high even with the θ_R correction.

2. Method II results are low for both porous materials. The same curves were obtained with Method II when (a) the entire θ range was considered (technique B) and (b) the partial θ range was used without the θ_R correction. Application of the θ_R correction by technique A did not

improve the fit between calculated and experimental values but increased the error. Conductivities calculated by Method II with $p = 2$ (not shown) were much too low.

3. Method III values are close to experimental values for both glass beads and loam soil. Neither the value of p nor the θ_R correction affected the results calculated by this method. Method III and Method I give identical results when residual water is accounted for and $p = 2$.

Conclusion

Method III introduced here circumvents problems inherent in Method I (Marshall's equation with matching factor) and Method II (Millington and Quirk's equation with matching factor). The results indicate the reliability of both Method I and Method III but indicate Method II was less accurate than the other methods. Both Methods I and II results are influenced by the arbitrary value of p, the exponent on the porosity term, while Method III results are unaffected by the choice of p. A correction for residual water is necessary for Method I, may increase calculation error in Method II (depending on the θ range used), and is unnecessary for Method III.

The computer program used for the calculations is available from either author.

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Table 1. Variations in the Conductivity Equation (equation (1)) For Three Methods of Computation

Method	Definition of ϵ	Definition of n	Value of p in equation (1) in previous work	Reference
I (Marshall)	θ; variable.	Number of water-filled pore classes corresponding to θ; variable.	$p = 2$	Marshall (1958) Nielsen et al. (1960) Jackson et al. (1965)
II (Millington and Quirk)	θ; variable.	Total number of pore-classes at saturation; constant.	$p = 1.0$ $p = 1.33$	Kunze et al. (1968) Millington and Quirk (1959) Jackson et al. (1965)
			$p = 2.0$	Sharma (1966)
III (this paper)	θ_s; constant.	Total number of pore-classes at saturation; constant.	-	-

Table 2. The Two Techniques Used to Account for Residual Water in Hydraulic Conductivity Calculations

Calculation Method	A: Using Partial θ Range With $n = m$ [1]	B: Using Entire θ Range With $n = m + n_R$ [2]
I and II	$\epsilon = \theta - \theta_R$	$\epsilon = \theta$
III	$\epsilon = \theta_s - \theta_R$	$\epsilon = \theta_s$

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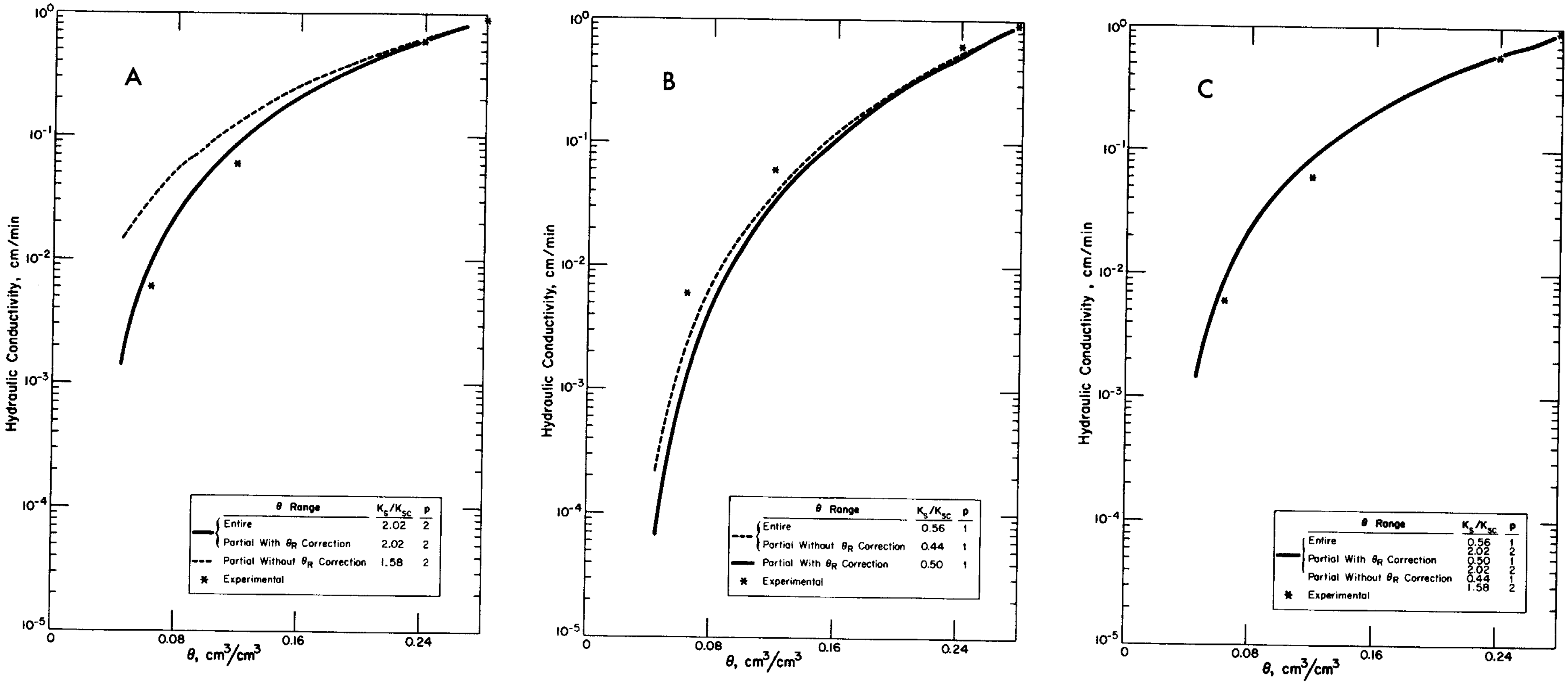
[1] m is the number of pore classes associated with the partial θ range used in the calculations. m is a variable in I and a constant in II.

[2] n_R is the number of pore classes associated with the range of water contents specified by θ_R, the residual water content.

FIGURE LEGENDS

Figure 1. Comparison of calculated hydraulic conductivities for monodispersed glass beads using Method I (Figure 1a), Method II (Figure 1b), and Method III (Figure 1c) with experimental values from Topp and Miller (1966, their Figure 2c). The values of the parameters used in equation (1) were $n = 20$, $\theta_R = 0.032$ cm^3/cm^3 (where applicable), and p = values specified in the figure. The calculated and experimental data were matched at θ_s.

Figure 2. Comparison of calculated hydraulic conductivities for Guelph loam using Method I (Figure 2a), Method II (Figure 2b), and Method III (Figure 2c) with experimental "discharge" values from Elrick and Bowman (1964, their Figure 3). The values of the parameters used in equation (1) were $n = 20$, $\theta_R = 0.23$ cm^3/cm^3 (where applicable), and p = values specified in the figure. The calculated and experimental data were matched at θ_s.



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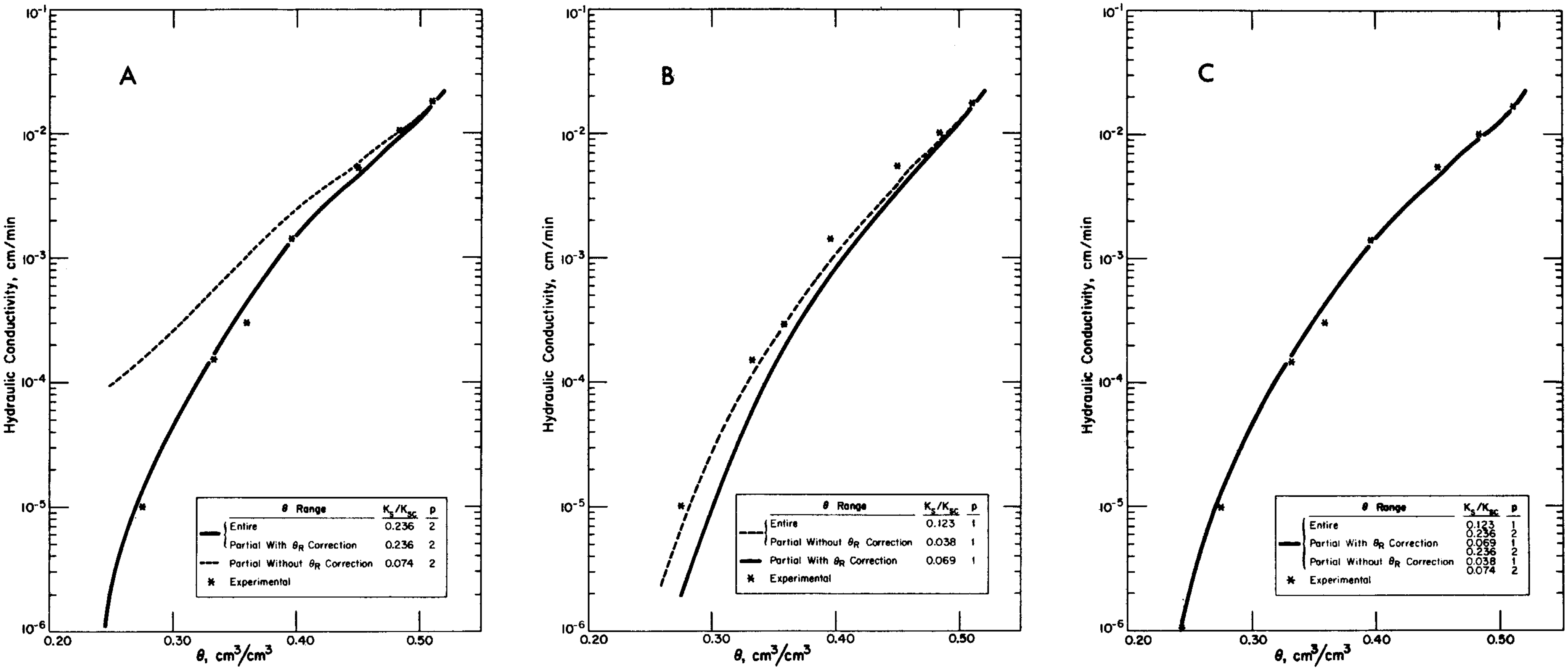

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