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ABSTRACT

A computer code has been developed to solve the groundwater flow equation in three dimensions. The code has finite-difference approximations solved by the strongly implicit solution procedure. Input parameters to the code include hydraulic conductivity, specific storage, porosity, accretion (recharge), and initial hydraulic head. These parameters may be input as varying spatially. The hydraulic conductivity may be input as isotropic or anisotropic. The boundaries either may permit flow across them or may be impermeable.

The code has been used to model leaky confined groundwater conditions and spherical flow to a continuous point sink, both of which have exact analytical solutions. The results generated by the computer code compare well with those of the analytical solutions. The code was designed to be used to model groundwater flow beneath fuel reprocessing and waste storage areas at the Savannah River Plant.

CONTENTS

DEVELOPMENT AND VALIDATION OF GWHEAD, A THREE-DIMENSIONAL GROUNDWATER HEAD COMPUTER CODE

INTRODUCTION

Radioactive liquid and solid wastes are stored above and below ground at the Savannah River Plant (SRP). Therefore, the Savannah River Laboratory (SRL) has developed quantitative models to predict the movement of potentially contaminated groundwater.

The objectives of the modeling program are

1) To develop a simulation of groundwater flow in the region of the subsurface underlying the general area of the waste storage facilities. This simulation is accomplished by determining the hydraulic head distribution over the study area and by calculating the flow velocity.

2) To simulate numerically potential radioactive waste movement in the ground. This simulation is a function of the groundwater velocity, hydrodynamic dispersion, ion exchange, and radioactive decay.

Both objectives depend upon laboratory and field investigations to obtain the necessary input data.

This report is arranged into two major sections: the theory and computer codes used to model groundwater flow and two sample applications of the computer codes which validate the accuracy of the computations by comparison to exact analytical solutions. In addition, an appendix describes the application of numerical analysis to the modeling problem.

MATHEMATICAL AND NUMERICAL MODELS

Modeling transient three-dimensional groundwater flow requires the solution of a differential mass continuity equation subject to several initial and boundary conditions. Fundamental to this modeling is Darcy's Law, which defines the groundwater flux as a function of hydraulic conductivity and groundwater potential. The relevant equations and conditions are discussed in the following sections.

Darcy's Law

Water movement in a saturated, porous medium can be predicted with Darcy's Law,

$$\vec{q} = -K\nabla\phi = n\vec{V} \qquad (1)$$

where

$\vec{q}$ = specific discharge or volume rate of flow through an arbitrary plane of the medium (L/T)

K = hydraulic conductivity (L/T)

ϕ = groundwater potential, piezometric head above an arbitrary datum (L)

n = porosity of the medium (dimensionless)

and

$\vec{V}$ = groundwater pore velocity (L/T)

By using the coordinate system of Figure 1, the potential, ϕ, is defined as

$$\phi = pg_c/\rho g + y_{SL} - y \qquad (2)$$

where

p = pressure at any point in the system (F/L^2)

ρ = density of water (M/L^3)

g = local acceleration due to gravity (L/T^2)

y_{SL} = Y coordinate of sea level (L)

y = Y coordinate of ϕ (L)

and

g_c = conversion factor (ML/FT^2)

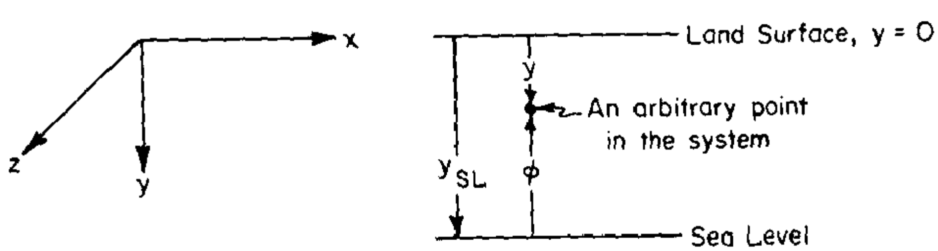

FIGURE 1. Coordinate System

Mass Continuity

The transient mass continuity equation for groundwater flow is

$$\vec{\nabla} \cdot (-n\vec{V}) + Q = S_s \frac{\partial \phi}{\partial t} \qquad (3)$$

where

Q = volumetric recharge or discharge rate per unit volume (1/T)

S_s = specific storage or the volume of water released from (or added to) storage per unit volume per unit decline (or rise) in head (1/L)

and

t = time (T)

Darcy's Law is substituted into the mass continuity equation and gives

$$\vec{\nabla} \cdot (K\vec{\nabla}\phi) + Q = S_s \frac{\partial \phi}{\partial t} \qquad (4)$$

Equation 4 is referred to as the groundwater flow equation and is used to obtain the distribution of ϕ in space and time. After ϕ has been found, Equation 1 is used to obtain the specific discharge $\vec{q}$ or the groundwater velocity $\vec{V}$.

Boundary Conditions for the Groundwater Flow Equation

Solution of the three-dimensional flow equation, Equation 4, requires six boundary conditions. The top surface represents the water table. The transient behavior of the water table is represented as a function of the dependent variable ϕ. The front, back, right, left, and bottom faces of the region of interest are assumed to have either flow or no-flow boundary conditions.

Free Surface Boundary Condition

A region of soil can be divided into two zones: 1) the saturated zone in which all pores are filled with water and 2) an overlying aerated zone in which the pores contain both air and water. The boundary between these two zones is the water table. The solution of Equation 4 in three dimensions is complicated by the presence of the water table (a free surface), because in the region above the water table, Equation 4 is not valid. The present model simulates only the water movement below the water table.

Because the water table separates the saturated and aerated zones, the water table is at atmospheric pressure; i.e., the water table is defined to occur wherever the water pressure $p = 0$. Therefore, from Equation 2, the water table is located where

$$\phi = y_{SL} - y \tag{5}$$

If a particle on the water table surface is considered, and only vertical accretion is permitted, the free surface has been represented in Bear[1] by

$$\frac{\partial\phi}{\partial t} - \frac{1}{n}\left[K_x\left(\frac{\partial\phi}{\partial x}\right)^2 + K_y\left(\frac{\partial\phi}{\partial y}\right)^2 + K_z\left(\frac{\partial\phi}{\partial z}\right)^2 + N_y\frac{\partial\phi}{\partial y} + K_y\frac{\partial\phi}{\partial y} + N_y\right] = 0 \tag{6}$$

where

N_y = vertical accretion, the volumetric flow rate per unit area from the aerated zone into the saturated zone (L/T)

Equation 6 is a generalized representation of the free surface boundary condition as a function of the dependent variable ϕ. Equation 6 was linearized by Bear[1] to become

$$\frac{\partial\phi}{\partial t} - \frac{1}{n}\cdot\left[N_y\frac{\partial\phi}{\partial y} + K_y\frac{\partial\phi}{\partial y} + N_y\right] = 0 \tag{7}$$

Equation 7 can be written as

$$\frac{\partial\phi}{\partial y} = \frac{n\frac{\partial\phi}{\partial t} - N_y}{K_y + N_y} \tag{8}$$

At the water table, Equation 8 is used in the groundwater flow equation. The linearization of Equation 6 assumes that the head gradients are relatively small; therefore, the accuracy of the head calculation will be best where there is minimal change in gradient.

No Flow Across a Boundary

A zero flux boundary is treated by assigning a zero conductivity to the boundary. With zero conductivity at the boundary, the flux by Darcy's Law must consequently be zero.

Flow Across a Boundary

Along a boundary which has water flow across it, the flux (specific discharge) is assumed to be a constant in the normal direction. For example, at the left or right face

$$\frac{\partial q_x}{\partial x} = 0 \tag{9}$$

Equation 9 allows the head outside the problem domain to be defined as a function of the head inside the domain. The gradient of the head existing between the last two grid blocks inside the domain is extrapolated linearly across the boundary. Most computer codes require specifying the boundaries as impermeable; the model space must, therefore, be made large enough so that these boundaries do not influence the solution. The extrapolation feature is intended to reduce the required size of the model space by more closely simulating the natural situation. However, the model space must still be specified large because the feature will produce errors in the solution unless head gradients are low; such low gradients generally occur at large distances from the major area of interest.

Initial Conditions

Solution of Equation 4 requires initial specification for ϕ in the problem domain. In addition, the water table position must be initially specified. The water table location governs the application of Equation 8, i.e., the free surface boundary of the problem domain.

Numerical Model

The numerical approach is to: 1) subdivide the region of interest into blocks in which the medium properties are assumed to be uniform, 2) replace derivatives with finite difference approximations, and 3) use a strongly implicit procedure[2,3] (SIP) to solve the resulting simultaneous difference equations.

Equation 4 is solved to give discrete values for the groundwater head as a function of space and time. The groundwater velocity components are then computed by using Darcy's Law.

Input required for the numerical solution is specific storage, hydraulic conductivity (isotropic or anisotropic), and porosity. Also required are the initial head distribution and the initial water table location.

MODEL VALIDATION

Two pumping situations for which exact analytical solutions exist have been simulated on the computer to illustrate the accuracy of the computational model. The leaky confined aquifer case presented is representative of some general groundwater conditions encountered in the field. The spherical flow case tests the ability of the computer code to model flow in three dimensions. Although exact analytical solutions exist for other flow situations (e.g., non-leaky confined aquifers), they provide no additional demonstration of the accuracy of the computational model and are, therefore, not presented.

Leaky Confined Aquifer

A common situation encountered in groundwater hydrology is that of flow to a pumping well that penetrates a confined aquifer (Figure 2). In this case, groundwater flows radially to the well, and water levels in the aquifer are drawn down to form a "cone of depression" with the apex of the inverted cone at the pumping well.

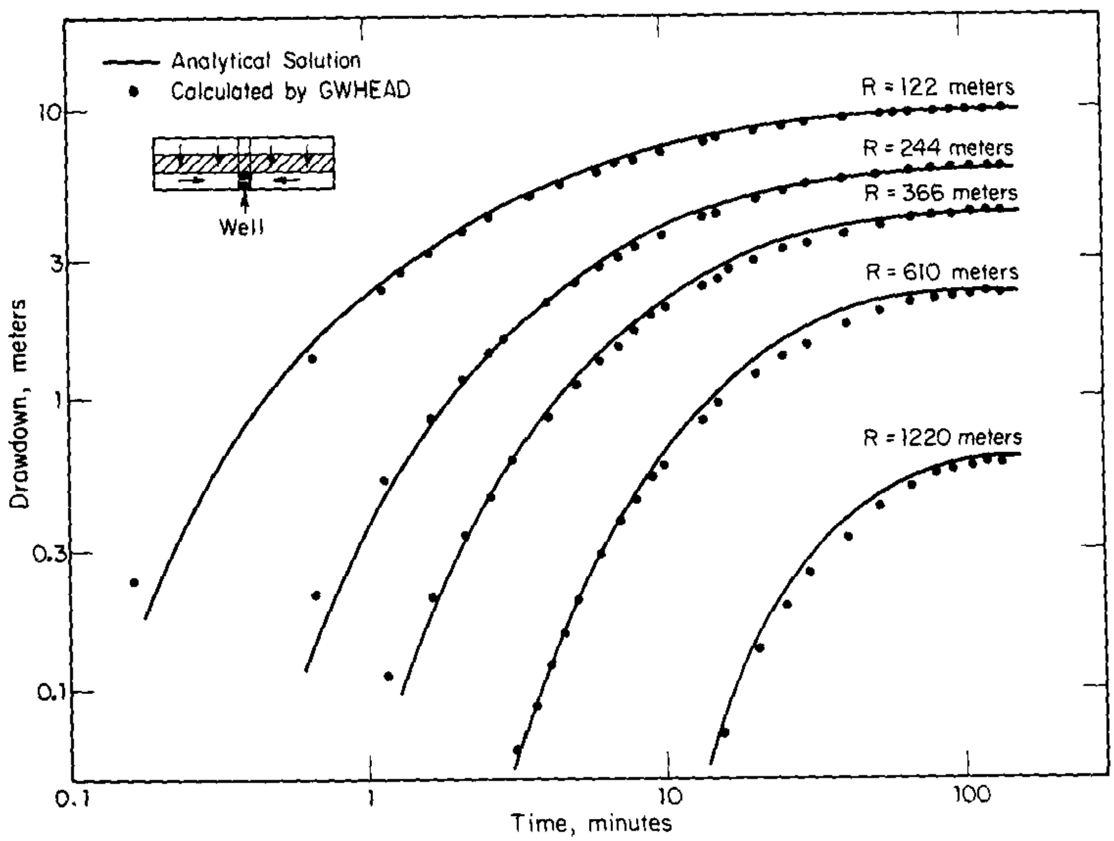

FIGURE 2. Comparison of Analytical and Numerical Calculations of Drawdown During Transient Leaky Flow

C. V. Theis[4] derived an analytical solution to the above case that yields the drawdown (s) of water levels in the aquifer at any distance (R) from the pumping well as a function of time. His solution is based upon several assumptions:

1) The well penetrates the full aquifer thickness and pumps at a constant rate.

2) The aquifer is homogeneous and isotropic in properties and infinite in areal extent.

3) The confining layer is relatively impermeable.

4) The coefficient of storage is constant and water is released from storage instantaneously with a decline in head.

For the present case, Theis' solution was also generalized by Hantush and Jacob[5] to include leakage of water downward through the confining layer. The solution for the leaky confined aquifer case is

$$s = \frac{Q}{4\pi T}\left[2K_o(2v) - \int_{\frac{v^2}{u}}^{\infty} \frac{1}{y}\exp\left(\frac{-y-v^2}{y}\right) dy\right]$$

where

s = drawdown (L)

Q = discharge (L^3/T)

T = transmissivity (L^2/T)

$u = R^2 S/(4Tt)$

$v = R/2 \sqrt{K'/b'T}$

y = variable of integration

S = storage coefficient (dimensionless)

K_o = modified Bessel function of the second kind and zero order

t = time (T)

R = distance from source or sink (L)

K' = vertical hydraulic conductivity of confining bed (L/T)

b' = thickness of the confining bed (L)

m = aquifer thickness (L)

The numerical simulation was set up with 1521 horizontal blocks in a 39 × 39 block arrangement. Each horizontal block was 122 meters on a side. The model was three layers deep vertically, with each layer 31 meters thick. The middle layer was assigned a hydraulic conductivity 400 times less than the top and bottom layers. The specific storage was the same throughout (3×10^{-6} per meter). Pumping was from the center block of the bottom layer. The time step was constant at 10 seconds, and the model was allowed to execute until steady state was achieved. Other parameters specified were porosity (15%), accretion (no accretion allowed), and the initial hydraulic head distribution (88 meters above sea level).

The drawdown versus time results at several values of R are shown in Figure 2. The numerical results compare favorably with those of the analytical solution. Accuracy of the solution is enhanced by reducing the horizontal size of the grid blocks and by reducing the size of the time steps. However, the improvement becomes less significant as the blocks approach a very small size (e.g., 10 meters). When the block size or time step size for each simultion is halved, about a 10% increase in accuracy is produced. Of course, increases in accuracy must be balanced against increases in execution time.

Spherical Flow

As an additional test to verify the numerical model's ability to simulate three-dimensional flow, an example was run to model flow to a continuous point sink in a semi-infinite isotropic media, as described by Narasimhan and Witherspoon.[6] The analytical solution is that of Carslaw and Jaeger[7] for the analogous heat flow situation:

$$s = \frac{w}{4\pi KR} \operatorname{erfc}\left(\frac{R}{(4Kt)^{\frac{1}{2}}}\right)$$

where

w = source of sink (L^4/T)

K = hydraulic diffusivity (L^2/T)

The problem was set up with 12,167 blocks in a 23 × 23 × 23 block cubical arrangement. Each block was 1.2 meters on a side.

The sink was at the center block of the cubical arrangement. The time step was constant at 1000 seconds for the entire period of the simulation. The hydraulic conductivity (3×10^{-6} meters per second), specific storage (3.6×10^{-3} per meter), and porosity (10%) were the same throughout the model. Also specified were accretion (no accretion permitted) and the initial hydraulic head distribution (259 meters).

The drawdown versus time results at several values of R are shown in Figure 3. The numerical results compare favorably with those of the analytical solution. Improved accuracy by decreasing block and time step sizes is possible for this case, as is discussed for the leaky confined aquifer.

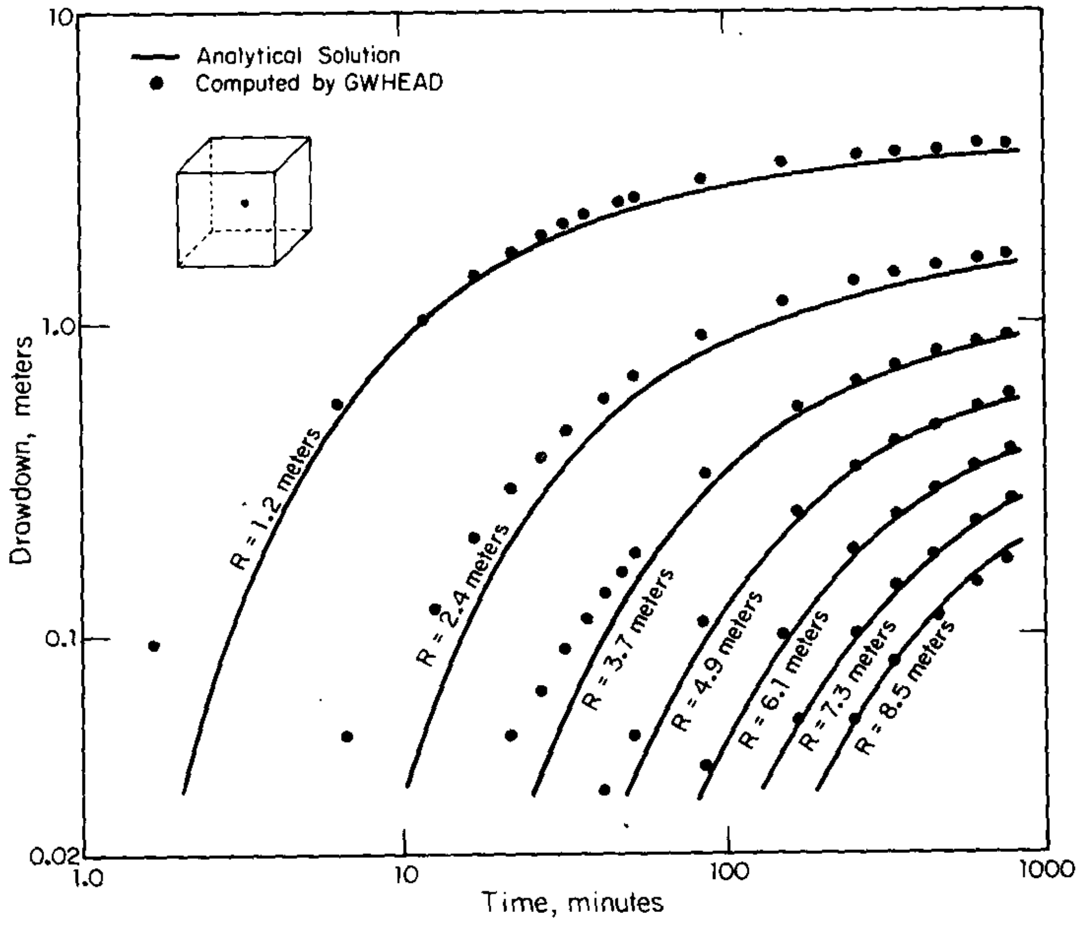

FIGURE 3. Comparison of Analytical and Numerical Calculations of Drawdown During Spherical Flow

APPENDIX

NUMERICAL ANALYSIS FOR GROUNDWATER FLOW EQUATION

The computer codes developed at SRL to simulate the three-dimensional movement of potential pollutants in the soil have been developed as a JOSHUA[8] subsystem. This subsystem, called GWATS (Ground Water And Transport Subsystem), consists of three major modules: GWHEAD, VELO, and TRANS.

GWHEAD computes the hydraulic head distribution in space and time. After the head distribution has been determined, the groundwater velocity components are computed by VELO. The resulting groundwater velocity field is used in the solution of the transport model TRANS.

Modules GWHEAD and VELO are currently operational. Module TRANS is being developed.

This appendix emphasizes the numerical analysis underlying the GWHEAD and VELO modules in the GWATS subsystem.

The numerical approach for the model is to 1) subdivide the region of interest into blocks in which the medium properties are assumed to be uniform, 2) replace derivatives in the governing geohydrological equations with finite difference approximations, and 3) use SIP, the strongly implicit procedure originally described by Stone,[2] to solve the simultaneous difference equations.

COORDINATE SYSTEM

The Cartesian coordinate system shown in Figure 1 is coded as shown in Figure A-1.

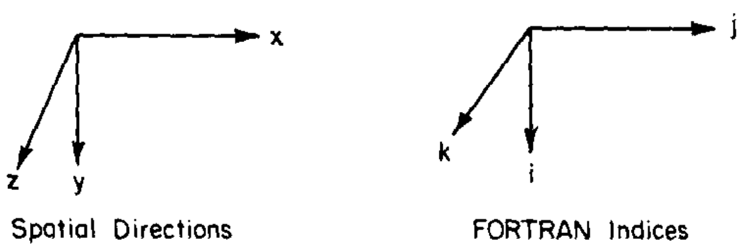

FIGURE A-1. Coordinate Indices

DARCY'S LAW

Conservation of Mass for Flow in the Horizontal Direction at the Water Table

The water table surface in reality may assume any shape. In the numerical model, the water table is approximated as shown in Figure A-2.

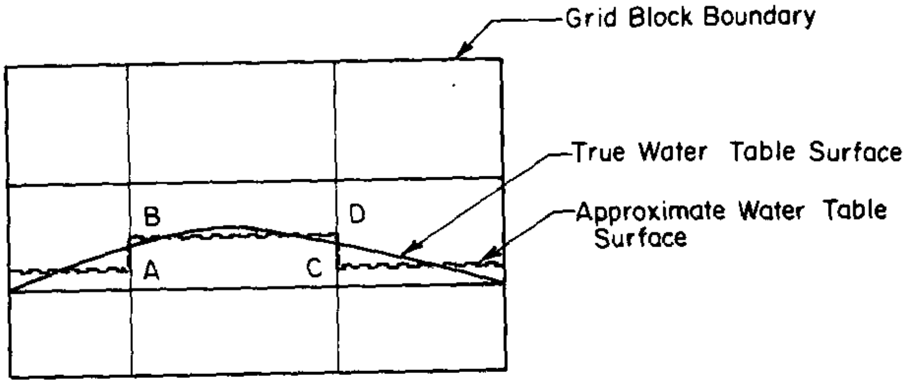

FIGURE A-2. Approximate Water Table

Because adjacent grid blocks may have different water levels, the following procedure is used to ensure mass conservation for horizontal flow across block faces at the water table.

Consider two adjacent blocks located at the water table surface as shown in Figure A-3. Let ϕ^* be the potential at the interface between the two blocks, and let $f_{i,j,k}\Delta y_i$ and $f_{i,j+1,k}\Delta y_i$ represent the water level in the jth and (j+1)th blocks, respectively (f is the fraction of the grid block containing water). Then the mass leaving the right face of the jth block is

$$m_{i,j,1} = -\rho_{i,j,k} K_{i,j,k} \left.\frac{\partial\phi}{\partial x}\right|_j f_{i,j,k}\Delta y_i \Delta z_k \Delta t \qquad \text{(A-1)}$$

where

$$\left.\frac{\partial\phi}{\partial x}\right|_j \cong \frac{\phi^* - \phi_{i,j,k}}{\Delta x_j/2} \qquad \text{(A-2)}$$

The mass entering the left face of the (j+1)th block is

$$m_{i,j+1,k} = -\rho_{i,j+1,k} K_{i,j+1,k} \left.\frac{\partial\phi}{\partial x}\right|_{j+1} f_{i,j+1,k}\Delta y_i \Delta z_k \Delta t \qquad \text{(A-3)}$$

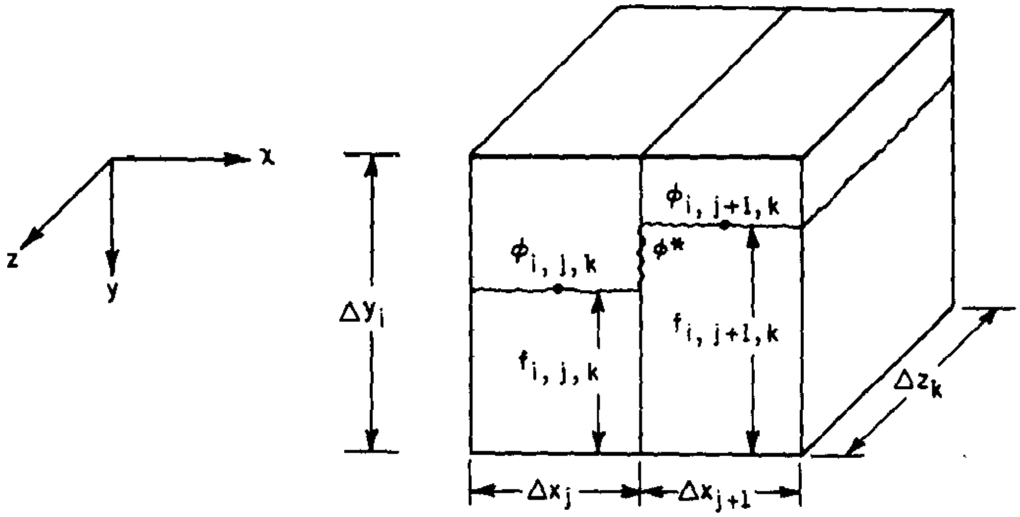

FIGURE A-3. Mass Conservation at the Water Table

where

$$\left.\frac{\partial\phi}{\partial x}\right|_{j+1} \cong \frac{\phi_{i,j+1,k} - \phi^*}{\Delta x_{j+1}/2} \quad \text{(A-4)}$$

Equating A-1 and A-3 gives

$$\phi^* = \frac{\dfrac{K_{i,j,k}\, f_{i,j,k}}{\Delta x_j}\,\phi_{i,j,k} + \dfrac{K_{i,j+1,k}\, f_{i,j+1,k}}{\Delta x_{j+1}}\,\phi_{i,j+1,k}}{\dfrac{K_{i,j,k}\, f_{i,j,k}}{\Delta x_j} + \dfrac{K_{i,j+1,k}\, f_{i,j+1,k}}{\Delta x_{j+1}}} \quad \text{(A-5)}$$

To represent the mass crossing the interface between jth and (j+1)th blocks in terms of $\phi_{i,j,k}$ and $\phi_{i,j+1,k}$, let

$$\rho_x q_x \,\Delta y_i \Delta z_k \Delta t = m_{i,j,k} \quad \text{(A-6)}$$

Equation A-5 substituted into A-1 and A-1 substituted into A-6 with $\rho_x = \rho_{i,j,k}$ yields

$$q_x \equiv q_{j,j+1} = \frac{-2K_{i,j,k}f_{i,j,k}K_{i,j+1,k}f_{i,j+1,k}\left[\phi_{i,j+1,k}-\phi_{i,j,k}\right]}{K_{i,j,k}f_{i,j,k}\Delta x_{j+1} + K_{i,j+1,k}f_{i,j+1,k}\Delta x_j} \tag{A-7}$$

where

$q_{j,j+1}$ = specific discharge from block j to (j+1).

Equation A-7 represents a conservative form of Darcy's Law at the interface between blocks j and (j+1) and is valid at or below the water table for flow in the x direction. A similar equation can be derived for flow in the z direction.

Conservation of Mass for Flow in the Vertical Direction at the Water Table

Two cases arise for flow in the y direction. The first is flow between grid blocks which are both full; this leads to an analog of Equation A-7 with all the $f_{i,j,k} = f_{i+1,j,k} = 1.0$. The second is that the blocks may not be full at the water table. In this latter case, conservation of mass requires that

$$\rho_{i,j,k}f_{i,j,k}K_{i,j,k}\left.\frac{\partial\phi}{\partial y}\right|_i \Delta x_j \Delta z_k \Delta t$$

$$= -\rho_{i+1,j,k}f_{i+1,j,k}K_{i+1,j,k}\left.\frac{\partial\phi}{\partial y}\right|_{i+1} \Delta x_j \Delta z_k \Delta t \tag{A-8}$$

where (see Figure A-4)

$$\left.\frac{\partial\phi}{\partial y}\right|_i \cong \frac{\phi^* - \phi_{i,j,k}}{f_{i,j,k}\Delta y_i} \tag{A-9}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{i+1} \cong \frac{\phi_{i+1} - \phi^*}{\Delta y_{i+1}/2} \tag{A-10}$$

and

$$f_{i+1,j,k} = 1.0 \tag{A-11}$$

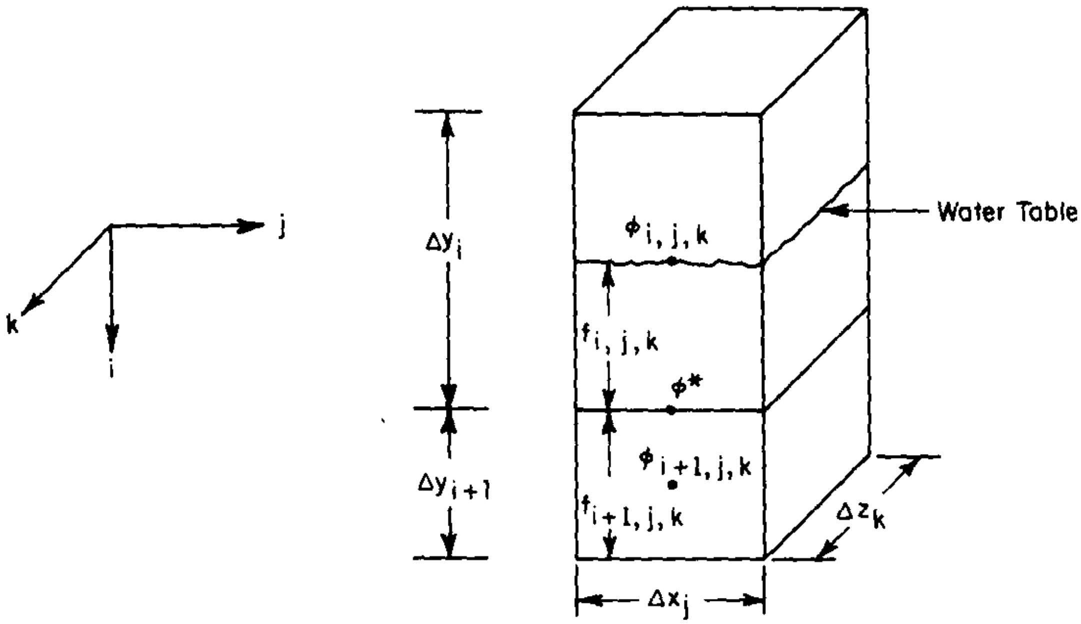

FIGURE A-4. Flow in the Y-Direction at the Water Table

Assuming $\rho_{i,j,k} = \rho_{i+1,j,k}$ leads to

$$q^{BWT}_{i,i+1} = \frac{-2K_{i,j,k}K_{i+1,j,k}\left(\phi_{i+1,j,k} - \phi_{i,j,k}\right)}{K_{i,j,k}\Delta y_{i+1} + 2K_{i+1,j,k}f_{i,j,k}\Delta y_i} \qquad \text{(A-12)}$$

where

$q^{BWT}_{i,i+1}$ = specific discharge across the bottom of a water table block (A-13)

Equation A-12 represents a conservative form of Darcy's Law for flow in the y direction at the water table.

GROUNDWATER FLOW EQUATION

The derivatives in the groundwater flow equation, Equation 4, can be approximated by using a Taylor series and an analog of Equation A-7. The finite difference form of the groundwater flow equation follows:

$$Z^n_{i,j,k}\phi^{n+1}_{i,j,k-1} + B^n_{i,j,k}\phi^{n+1}_{i,j-1,k} + D^n_{i,j,k}\phi^{n+1}_{i-1,j,k}$$

$$+ E^n_{i,j,k}\phi^{n+1}_{i,j,k} + F^n_{i,j,k}\phi^{n+1}_{i+1,j,k} + H^n_{i,j,k}\phi^{n+1}_{i,j+1,k}$$

$$+ S^n_{i,j,k}\phi^{n+1}_{i,j,k+1} = W^n_{i,j,k} \qquad \text{(A-14)}$$

$$i = 1,2,\ldots,IX$$

$$j = 1,2,\ldots,JX$$

$$k = 1,2,\ldots,KX$$

where

$$Z^n_{i,j,k} = \frac{2K_{i,j,k}f^n_{i,j,k}K_{i,j,k-1}f^n_{i,k,k-1}}{\left[K_{i,j,k}f^n_{i,j,k}\Delta z_{k-1} + K_{i,j,k-1}f^n_{i,j,k-1}\Delta z_k\right]\Delta z_k} \qquad \text{(A-15)}$$

$$B^n_{i,j,k} = \frac{2K_{i,j,k}f^n_{i,j,k}K_{i,j-1,k}f^n_{i,j-1,k}}{\left[K_{i,j,k}f^n_{i,j,k}\Delta x_{j-1} + K_{i,j-1,k}f^n_{i,j-1,k}\Delta x_j\right]\Delta x_j} \qquad \text{(A-16)}$$

$$D^n_{i,j,k} = \frac{2K_{i,j,k}f^n_{i,j,k}K_{i-1,j,k}f^n_{i-1,j,k}}{\left[K_{i,j,k}f^n_{i,j,k}\Delta y_{i-1} + K_{i-1,j,k}f^n_{i-1,j,k}\Delta y_i\right]\Delta y_i} \qquad \text{(A-17)}$$

$$E^n_{i,j,k} = -\left[Z^n_{i,j,k} + B^n_{i,j,k} + D^n_{i,j,k} + F^n_{i,j,k}\right.$$

$$\left. + H^n_{i,j,k} + S^n_{i,j,k}\right] - \frac{S^n_{i,j,k}}{\Delta t} \qquad \text{(A-18)}$$

$$F^n_{i,j,k} = \frac{2K_{i,j,k}f^n_{i,j,k}K_{i+1,j,k}f^n_{i+1,j,k}}{\left[K_{i,j,k}f^n_{i,j,k}\Delta y_{i+1} + K_{i+1,j,k}f^n_{i+1,j,k}\Delta y_i\right]\Delta y_i} \qquad \text{(A-19)}$$

$$H^n_{i,j,k} = \frac{2K_{i,j,k}f^n_{i,j,k}K_{i,j+1,k}f^n_{i,j+1,k}}{\left[K_{i,j,k}f^n_{i,j,k}\Delta x_{j+1} + K_{i,j+1,k}f^n_{i,j+1,k}\Delta x_j\right]\Delta x_j} \qquad \text{(A-20)}$$

$$S^n_{i,j,k} = \frac{2K_{i,j,k} f^n_{i,j,k} K_{i,j,k+1} f^n_{i,j,k+1}}{K_{i,j,k} f^n_{i,j,k} \Delta z_{k+1} + K_{i,j,k+1} f^n_{i,j,k+1} \Delta z_k} \qquad \text{(A-21)}$$

$$W^n_{i,j,k} = - Q_{i,j,k} - \frac{S^n_{i,j,k} \phi^n_{i,j,k}}{\Delta t} \qquad \text{(A-22)}$$

and

n = time step index.

It should be noted that the fractions $f^n_{i,j,k}$ make all the coefficients in Equation A-14 time dependent. In the solution for $\phi^{n+1}_{i,j,k}$, these coefficients are all based upon $f^n_{i,j,k}$, i.e., previous time step values.

Equation A-14 leads to (IX)*(JX)*(KX) linear algebraic equations in the unknowns $\phi^{n+1}_{i,j,k}$. This system of equations is solved iteratively with the SIP algorithm.

DETERMINATION OF THE TRANSIENT WATER TABLE LOCATION

An initial water table distribution in space is specified, and the water table position as a function of time is determined by first solving the groundwater flow equation for ϕ and then applying Equation 2:

$$\phi(x,y,z,t_2) = y_{SL} - y(t_2) \qquad \text{(A-23)}$$

and

$$\phi(x,y,z,t_1) = y_{SL} - y(t_1) \qquad \text{(A-24)}$$

Equation A-24 subtracted from Equation A-23 yields

$$y(t_2) = y(t_1) - [\phi(x,y,z,t_2) - \phi(x,y,z,t_1)] \qquad \text{(A-25)}$$

where $y(t_2)$ is the new water table position for coordinates x,y,z.

NO FLOW ACROSS A BOUNDARY

Zero flow across a boundary is accomplished by specifying a zero conductivity for nodes at the boundary. For example, at the right-hand boundary

$$K_{i,JX+1,k} = 0 \tag{A-26}$$

where JX is the upper limit of the index j. This condition can be implemented in a similar fashion at other boundaries.

FLOW ACROSS A BOUNDARY

Flow across a boundary, for example the right-hand boundary, can be simulated by making the approximation (see Figure A-5)

$$q_{JX-1,JX} = q_{JX,JX+1} \tag{A-27}$$

Using Equation A-7 and assuming

$$K_{i,JX+1,k} = K_{i,JX,k} \tag{A-28}$$

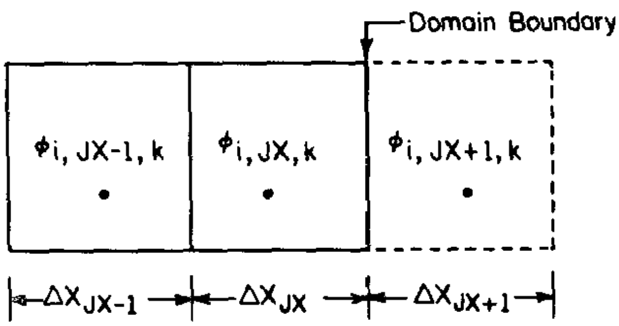

FIGURE A-5. Boundary Flow

$$f^n_{i,JX+1,k} = f^n_{i,JX,k} \tag{A-29}$$

and

$$\Delta x_{JX+1} = \Delta x_{JX} \tag{A-30}$$

Equation A-27 becomes

$$\frac{-2K_{i,JX-1,k}f^n_{i,JX-1,k}K_{i,JX,k}f^n_{i,JX,k}\left(\phi^{n+1}_{i,JX,k} - \phi^{n+1}_{i,JX-1,k}\right)}{K_{i,JX-1,k}f^n_{i,JX-1,k}\Delta x_{JX} + K_{i,JX,k}f^n_{i,JX,k}\Delta x_{JX-1}}$$

$$= \frac{-K_{i,JX,k}f^n_{i,JX,k}\left(\phi^{n+1}_{i,JX+1,k} - \phi^{n+1}_{i,JX,k}\right)}{\Delta x_{JX}} \tag{A-31}$$

Setting

$$D^{*,n}_{i,JX,k} = \frac{K_{i,JX,k} f^{n}_{i,JX,k}}{\Delta x^{2}_{JX}} \qquad \text{(A-32)}$$

Equation A-16 and Equation A-31 yield

$$\phi^{n+1}_{i,JX+1,k} = \frac{B^{n}_{i,JX,k}}{D^{*,n}_{i,JX,k}} \left(\phi^{n+1}_{i,JX,k} - \phi^{n+1}_{i,JX-1,k}\right) + \phi^{n+1}_{i,JX,k} \qquad \text{(A-33)}$$

Equation A-33 expresses the potential outside the problem domain as a function of potentials inside the domain.

DETERMINATION OF THE POTENTIAL WHEN A RISING WATER TABLE CROSSES A BLOCK BOUNDARY

Assume that at time $t = t_1$, the water table is in block (i,j,k), and at time $t = t_2$, the water table rises into block (i-1,j,k) as shown in Figure A-6. Then for a rising water table

$$y_{SL} - \sum_{\ell=1}^{i-2} \Delta y_{\ell} > \phi^{n+1}_{i,j,k} > y_{SL} - \sum_{\ell+1}^{i-1} \Delta y_{\ell} \qquad \text{(A-34)}$$

Therefore, $\phi^{n+1}_{i-1,j,k}$ is set equal to the newly calculated value of $\phi^{n+1}_{i,k,k}$ which leaves $\phi^{n+1}_{i,j,k}$ to be defined.

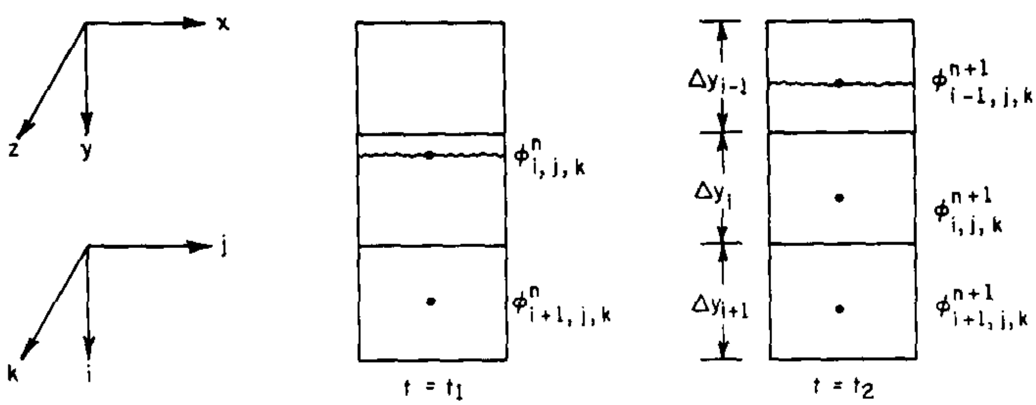

FIGURE A-6. Rising Water Table

. If the pressure varies linearly from block (i+1,j,k) to block (i-1,j,k), Equation 2 may be written as

$$P^{n+1}_{i+1,j,k} = \frac{\rho g}{144 g_c}\left[\phi^{n+1}_{i+1,j,k} - \left(y_{SL} - \sum_{\ell=1}^{i} \Delta y_\ell - \Delta y_{i+1}/2\right)\right] \qquad \text{(A-35)}$$

From the definition of the water table

$$P^{n+1}_{i-1,j,k} = 0 \qquad \text{(A-36)}$$

and Equation A-35, interpolation yields

$$P^{n+1}_{i,j,k} = \frac{\rho g}{g_c}\left[\phi^{n+1}_{i+1,j,k} - y_{SL} + \sum_{\ell=1}^{i} \Delta y_\ell + \frac{1}{2}\Delta y_{i+1}\right]$$

$$\left[\frac{1}{2}\Delta y_i + \sum_{\ell=1}^{i-1} f^{n+1}_{\ell,j,k}\,\Delta y_\ell\right] \Big/$$

$$\left[\frac{1}{2}\Delta y_{i+1} + \Delta y_i + f^{n+1}_{i-1,j,k}\,\Delta y_{i-1}\right] \qquad \text{(A-37)}$$

Equation A-37 and Equation 2 yield

$$\phi^{n+1}_{i,j,k} = \left[\phi^{n+1}_{i+1,j,k} - y_{SL} + \sum_{\ell=1}^{i} \Delta y_\ell + \frac{1}{2}\Delta y_{i+1}\right] \times$$

$$\left[\frac{1}{2}\Delta y_i + \sum_{\ell=1}^{i-1} f^{n+1}_{\ell,j,k}\,\Delta y_\ell\right] \Big/$$

$$\left[\frac{1}{2}\Delta y_{i+1} + \Delta y_i + f^{n+1}_{i-1,j,k}\,\Delta y_{i-1}\right]$$

$$+ y_{SL} - \sum_{\ell=1}^{i-1} \Delta y_\ell - \frac{1}{2}\Delta y_i \qquad \text{(A-38)}$$

Equation A-38 is an approximation to ϕ in the block (i,j,k) immediately after the water table has risen above the top of the block.

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