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QUANTITATIVE SAFETY ANALYSIS – II

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ABSTRACT

Several examples of probabilistic models often used to analyze fault trees, common mode failures, and decision trees are discussed to show that each is a special case of a general model. The general model is PAR (Probabilistic Analysis of Risks), and its use enhances the understanding and interpretation of this type of analysis. Experience with this method of analysis over the past few years has shown that the generality originally incorporated in the method may have obscured its real usefulness. Many types of real life situations (fault trees, event trees, decision trees, and common mode failures) are special cases of the PAR method. The numerical answer that results from the analysis of any one of these special cases is not likely to be as accurate as a single number implies. The content and extent of the analysis are more important than the single numerical answer. A broad range of meaning, significance, and implications can be reviewed by examining these problems through the generality of the PAR method. The PAR method was intentionally designed to provide a flexibility that would assist in easing the task of assembling and reviewing the input data, especially in those cases where the data must reflect primarily judgment factors. Greater generality will hopefully prevent the false sense of security provided by a single numerical answer and a restricted view of the problem.

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INTRODUCTION

The Research and Development Report, DP-1207, *Quantitative Safety Analysis*, J. W. Croach and L. M. Arnett, April 1970, presented an analytical method and a computer program to assess the safety of an operation or of a mission.[1] Although the Probabilistic Analysis of Risk (PAR) method was recognized as being of considerable generality, some of the specific embodiments and special cases were not clearly delineated for the reader of the earlier report. This report is an attempt to present some further explanations and clarifications that have resulted from experience with the method.

It is believed that the more general the framework of the analysis, the more likely will the analyst understand the results of the procedure. The restriction of attention to special cases may well produce valid numerical results, but some of the implications, extensions, and interpretations may be missed.

Although the detailed examination of how and why the PAR method and its computer program works seems involved and intricate, the use of the procedures and the interpretation of the results are disarmingly routine. A little experience with this method soon reveals that the consideration of real probelms results in rather simple and easily understood logical structures. Various sets of restrictions on variables in the PAR method produce the special cases such as fault trees, event trees, decision trees, and common mode failures.

The probabilistic problem is specified by listing a set of n declarative statements including one statement whose probability of truth or falsity answers the question of concern. The set of n declarative statements are ordered from 1 to n. The order is completely arbitrary, but in an actual situation the order will be found to be very natural. The following set of conditional probabilities including the unconditional probability, P(1), defines the entire problem:

$P(1)$

$P(2|1)$

$P(2|\bar{1})$

$P(3|2,1)$

$P(3|2,\bar{1})$

$P(3|\bar{2},1)$

$P(3|\bar{2},\bar{1})$

.

.

.

$P(k|k-1,\ k-2,\ \ldots\ 2,1)$

$P(k|k-1,\ k-2,\ \ldots\ 2,\bar{1})$

.

.

.

$P(n|n-1,\ n-2,\ \ldots\ 2,1)$

$P(n|n-1,\ n-2,\ \ldots\ 2,\bar{1})$

$(n|n-1,\ n-2,\ \ldots\ 2,\bar{1})$

.

.

.

$P(n|\overline{n-1},\ \overline{n-2},\ \ldots\ \bar{2},\bar{1})$

The input for the PAR computer program does not require any probabilities that equal zero. The large number of independencies and zero probabilities that are encountered naturally in real problems reduces most problems to a size much smaller than that implied by the theoretical statement.

REVIEW OF PROBABILITY THEOREMS

This brief review of some theorems and axioms of probability is not complete nor necessarily rigorous, in a mathematical sense, but rather it contains some reminders of concepts that help to clarify the material that follows. All of the following material can be found in standard texts on probability (for example, Reference 2).

A probabilistic experiment is a prescribed procedure that results in one of several mutually exclusive and exhaustive outcomes. For example, when three coins are tossed and allowed to fall at random, each coin must show either a head or a tail. This experiment can produce only eight outcomes when the three coins are distinguishable:

HHH, THH, HTH, HHT, TTH, THT, HTT, TTT

These outcomes are mutually exclusive because only one outcome can be the result of the experiment, and they are exhaustive because one of the outcomes in the list must be the result. The list of outcomes is said to constitute the sample space or the certain event. The list of outcomes is a set of outcomes. A subset of the sample space is an event. For n elements in the sample space, there are 2^n subsets or events when the empty set containing no elements is counted.

To each event or subset, a number is assigned in the closed interval zero to unity. This assigned number is called the probability. These numbers must satisfy three axioms; viz.

- Axiom I: $1 \geqslant P(e) \geqslant 0$
- Axiom II: $P(S) = 1$

  The probability of the sample space, S,

  which means some element in the space, is equal to unity.
- Axiom III: If $E_1 \cap E_2 = \emptyset$ (empty), then

$$P(E_1 + E_2) = P(E_1) + P(E_2)$$

If events E_1 and E_2 are mutually exclusive, i.e., the intersection of the two sets that define the events contain no elements in common, then the probability of E_1 or E_2 is the sum of the probabilities of E_1 and E_2 separately. It can be shown that all the theory of probability can be deduced from these three axioms.

A physical interpretation of probability is likely to be more meaningful or more effective in establishing the concept in one's technical thinking. If the given experiment were to be repeated a large number of times, eventually approaching infinity, then the probability of the event E is the ratio of the number of times the event E occurs, n_E, to the total number of repetitions, n_T, of the experiment:

$$P(E) = \lim_{n_T \to \infty} \frac{n_E}{n_T}$$

The selection of subsets to be considered as events is arbitrary and needs to be only sufficient to answer the questions in a specific instance. For example, when three coins are tossed and there are eight outcomes in the sample space, the interest may be only in how many heads are showing. Thus, there are only four events; viz.,

Event	*Outcomes*
0 - heads	TTT
1 - heads	HTT, THT, TTH
2 - heads	HHT, HTH, THH
3 - heads	HHH

There equally well might have been only two events if the interest were confined to some heads and no heads.

Sometimes two events may include outcomes in common. For example, when a die is rolled, the face showing may be both even, E_1, and larger than three, E_2. In set notation, this is shown by

$$E_1 = \{2,4,6\}$$

$$E_2 = \{4,5,6\}$$

Both the outcomes 4 and 6 appear in the subsets that define E_1 and E_2, and if either of these outcomes occurs, then both events occur. In this situation, the events are said to be joint. Otherwise, when the two events cannot both occur, then the two events are said to be disjoint, and

$$P(E_1E_2) = P(E_1 \cap E_2)$$

When two events are joint, i.e., they could both occur, the probability that they will both occur is defined as

$$P(E_1E_2) = P(E_2|E_1)\cdot P(E_1)$$
$$= P(E_1|E_2)\cdot P(E_2)$$

where $P(E_2|E_1)$ is the conditional probability of E_2 given that E_1 has occurred. These conditional probabilities are probabilities in a restricted sample space because it is given or known that a specified event has already happened. For example, the probability that the face of a die is greater than 3 given that the face is even is

$$P(>3|\text{even}) = \frac{P(>3 \text{ and even})}{P(\text{even})}$$
$$= \frac{2/6}{3/6} = \frac{2}{3}$$

The correctness of this result is readily seen by examining the structure of the subsets of the events:

$$\text{Even} = \{2,4,6\}$$
$$>3 = \{4,5,6\}$$

Two times in three trials the face will be greater than 3 given that the face is either 2, 4, or 6. This theorem can be extended to any number of events as follows:

$$P(E_1E_2E_3) = P(E_3|E_2E_1)\cdot P(E_2|E_1)\cdot P(E_1)$$
$$= P(E_2|E_3E_1)\cdot P(E_3|E_1)\cdot P(E_1)$$
$$= P(E_1|E_3E_2)\cdot P(E_3|E_2)\cdot P(E_2)$$
$$\vdots$$

The various relationships between joint and conditional probabilities can be summarized in the expressions:

$$P(X_iX_j) = P(X_i|X_j)\cdot P(X_j)$$
$$= P(X_j|X_i)\cdot P(X_i)$$
$$P(X_iX_j|X_k) = P(X_j|X_k)\cdot P(X_i|X_jX_k)$$
$$= P(X_i|X_k)\cdot P(X_j|X_iX_k)$$

X_i, X_j, and X_k represent sets such as {abc...i}, where a, b, c, ..., i are simple events.

By De Morgan's law, the complement of an intersection, $\overline{AB}$, is not equal to the intersection of the complements but rather is the union of all sets that contain at least one complementary set; e.g.

$$\{\overline{AB}\} = \{A\bar{B}\} \cap \{\bar{A}B\} \cap \{\bar{A}\bar{B}\}$$

From Axiom III,

$$P(\overline{AB}) \neq P(\bar{A}\bar{B})$$

$$= P(A\bar{B}) + P(\bar{A}B) + P(\bar{A}\bar{B})$$

Sometimes, the probability of an event is not conditional upon another event, i.e.

$$P(E_2 | E_1) = P(E_2)$$

When this is true, the two events are said to be independent, and may also be expressed by the probability that both will occur

$$P(E_2 E_1) = P(E_2) \cdot P(E_1)$$

When the experiment is the tossing of two coins, the probability that the second coin falls head is independent of the result of the first toss. With two coins, the probability of a head is

$$P(H) = 1/2, \text{ and}$$

$$P(HH) = 1/2 \cdot 1/2 = 1/4$$

However, when two cards are drawn, without replacement, from a deck of the usual 52 playing cards, the probabilities for the second drawing are conditional upon the result of the first drawing. For example, what is the probability that both cards will be hearts? The probability of a heart on the first draw is

$$P(H_1) = 13/52 = 1/4$$

There are 13 hearts in 52 cards, and each card has the same probability of being selected. Now what is the probability that the second card will be a heart given that the first card was a heart? Obviously, it is

$$P(H_2 | H_1) = 12/51$$

After the first card is drawn and found to be a heart, there are now only 12 hearts left in 51 cards. Finally

$$P(H_2H_1) = P(H_2|H_1) \cdot P(H_1)$$

$$= 12/51 \cdot 13/52 = 1/17$$

The two events are not independent because

$$P(H_2|H_1) \neq P(H_2)$$

It is important to note that events are dependent or independent in groups, e.g. pairwise. Dependency is meaningless when applied to a single event.

The independency criteria may be summarized as follows:

- If $P(abc\ldots n) = P(a) \cdot P(b) \cdot P(c)\ldots P(n)$ then the set $\{abc\ldots n\}$ is independent as a group.
- If $P(X_iX_j) = P(X_i) \cdot P(X_j)$ then any subset of X_i is independent of any subset of X_j.
- If the set $\{abc\ldots n\}$ is independent as a group, then they are all independent pairwise. However, the converse is not necessarily true. The group may be independent pairwise but dependent as a group.

In experiments whose outcomes consist of combinations of trials, as in the tossing of three coins with its eight outcomes, the event of interest may be the result of a specific trial. For example, the event may be head on the third toss. What is the probability of this event?

$$P(E) = P(H_1H_2H_3) + P(T_1H_2H_3) + P(H_1T_2H_3) + P(T_1T_2H_3)$$

where

$$E = \text{head on third toss}$$

The unconditional probability of the event is the sum of all the probabilities of the separate outcomes for which the event is true. In this instance, each outcome is equally likely and equals 1/8 so that $P(E) = 4 \cdot 1/8 = 1/2$ as expected. As pointed out earlier, each of the probabilities of the outcomes may be expressed in terms of conditional probabilities.

SOME ELEMENTARY MODELS

A simple or elementary probabilistic experiment might be one that possesses the following characteristics. Before the experiment, three statements might be formulated such that after the conclusion of the experiment the truth or falsity of each of the three statements could be determined. For example, the experiment might consist of drawing three cards from a deck, and the three statements might be

1. The first card is a heart.

2. The second card is a spade.

3. The third card is a club.

Surely, after the experiment is concluded, it is a simple matter to determine the truth or falsity of each of the three statements: nothing beyond these simple conditions is implied in defining a probabilistic experiment in these terms.

The eight outcomes may be designated as

123

$\bar{1}23$

$1\bar{2}3$

$12\bar{3}$

$\bar{1}\bar{2}3$

$\bar{1}2\bar{3}$

$1\bar{2}\bar{3}$

$\bar{1}\bar{2}\bar{3}$

where x stands for x being true, $\bar{x}$ stands for x being false. Because the eight outcomes are exhaustive, the sum of their probabilities must add to unity:

$$P(123) + P(\bar{1}23) + P(1\bar{2}3) + \dots = 1$$

Therefore, there are only seven independent probabilities, because given any seven probabilities, the eighth probability can be readily calculated. Everything of a probabilistic nature about this experiment can be answered, given the knowledge of the probabilities of any seven of these outcomes.

There are many other sets of seven independent probabilities from which the probabilities of the seven outcomes may be calculated. This situation becomes very advantageous if there exists some set of probabilities that is easier to obtain or estimate than the final set for the seven outcomes; e.g.

$P(1) = a$

$P(2|1) = b$

$P(2|\bar{1}) = c$

$P(3|21) = d$

$P(3|2\bar{1}) = e$

$P(3|\bar{2}1) = f$

$P(3|\bar{2}\bar{1}) = g$

are seven independent probabilities from which the probabilities of the seven outcomes listed above can be calculated.

The word "independent" is used in two senses in the preceding material. Independent probabilities mean arbitrary probabilities such that no subset of the given set is sufficient to determine the values of the whole set. This is the usual mathematical meaning. Independent events, however, are events such that the outcome of one trial does not affect the outcome of a succeeding trial.

There is no restriction other than

$0 \leqslant a, b, c, d, e, f, g \leqslant 1$

placed on the seven probabilities listed above. An example of the calculational procedure is

$P(123) = P(3|21) \cdot P(2|1) \cdot P(1)$

All of the probabilities of the eight outcomes can be calculated from the seven given probabilities:

$P(123) = a \cdot b \cdot d$

$P(\bar{1}23) = (1-a) \cdot c \cdot e$

$P(1\bar{2}3) = a \cdot (1-b) \cdot f$

$P(12\bar{3}) = a \cdot b \cdot (1-d)$

$$P(\bar{1}\bar{2}3) = (1-a)\cdot(1-c)\cdot g$$

$$P(\bar{1}2\bar{3}) = (1-a)\cdot c\cdot(1-e)$$

$$P(1\bar{2}\bar{3}) = a\cdot(1-b)\cdot(1-f)$$

$$P(\bar{1}\bar{2}\bar{3}) = (1-a)\cdot 1-c)\cdot(1-g)$$

The sum of these eight probabilities equals unity as required.

Nothing is implied about the independence relations among these three statements nor about the chronological sequence of the three statements. Although the original input to the problems contained $P(1) = a$, this does not imply that $P(1|3)$ is necessarily $P(1)$. In fact, in general $P(1|3)$ will be some finite value different from $P(1)$. This merely means that 1 and 3 are not independent. The practical implication of these observations is that the envisioned experiment is very general and thus can represent a tree, net, web, or chain depending upon the specific embodiment. This versatility will become more evident with further examples and discussion.

After all the probabilities of the eight exhaustive outcomes have been calculated, other probabilities of interest can be readily calculated as illustrated in the following examples:

$$P(3) = P(123) + P(\bar{1}23) + P(1\bar{2}3) + P(\bar{1}\bar{2}3)$$

$$P(2) = P(123) + P(\bar{1}23) + P(12\bar{3}) + P(\bar{1}2\bar{3})$$

$$P(23) = P(123) + P(\bar{1}23)$$

$$P(2|3) = \frac{P(23)}{P(3)} = \frac{P(123) + P(\bar{1}23)}{P(123) + P(\bar{1}23) + P(1\bar{2}3) + P(\bar{1}\bar{2}3)}$$

The list of possible probabilities becomes lengthy even for only three statements when all the possible joint and conditional probabilities are considered. Normally, very few of this large number of probabilities are of interest in a practical problem. In a particular instance in which one of these unusual possibilities is of concern, an additional statement can be added to the list of original statements to define any joint possibility. The conditional probabilities are each just a ratio of joint probabilities.

The great merit in specifying a problem in terms of conditional probabilities is that there is nothing absolute about the conditionalities. This means that no statement is true and all others are false, but rather that all statements are true and the convenient ones can be chosen for use; e.g.

$$P(c) = P(cb) + P(c\bar{b})$$

$$= P(c|b)\cdot P(b) + P(c|\bar{b})\cdot P(\bar{b})$$

$$P(c) = P(cba) + P(cb\bar{a}) + P(c\bar{b}a) + P(c\bar{b}\bar{a})$$

$$= P(c|ba)\cdot P(b|a)\cdot P(a) + P(c|b\bar{a})\cdot P(b|\bar{a})\cdot P(\bar{a})$$

$$+ P(c|\bar{b}a)\cdot P(\bar{b}|a)\cdot P(a) + P(c|\bar{b}\bar{a})\cdot P(\bar{b}|\bar{a})\cdot P(\bar{a})$$

are both correct and numerically the same. The choice in an actual problem is dictated merely by what is known about the systems. The relationship between independence and conditionality is an abstruse one and of no concern when the problem is stated entirely in terms of conditional probabilities.

The example of the three statements listed in the illustration of drawing three cards is interesting to carry further because, it is a case in which all three statements are dependent. The input for this problem is

$$P(1) = 13/52 = 1/4$$

$$P(2|1) = 13/51$$

$$P(2|\bar{1}) = 38/153$$

$$P(3|21) = 13/50$$

$$P(3|2\bar{1}) = 481/1900$$

$$P(3|\bar{2}1) = 481/1900$$

$$P(3|\bar{2}\bar{1}) = 1419/5750$$

When these values are substituted in the proper formulas, the results are as follows:

$$P(123) = 0.016569$$

$$P(\bar{1}23) = 0.047157$$

$$P(1\bar{2}3) = 0.047157$$

$$P(12\bar{3}) = 0.047157$$

$$P(\bar{1}\bar{2}3) = 0.139118$$

$$P(\bar{1}2\bar{3}) = 0.139118$$

$$P(1\bar{2}\bar{3}) = 0.139118$$

$$P(\bar{1}\bar{2}\bar{3}) = \underline{0.424608}$$

$$\text{Total} = 1.000002$$

From these results,

$P(1) = 0.250001$

$P(2) = 0.250001$

$P(3) = 0.250001$

These values should be exactly 0.25 and would be except for the round-off error. The unconditional values are just what one would expect, for no matter how many cards are drawn, there is no preference for selecting any one suit, so that the unconditional probability of the truth of each of the statements is just 1/4.

To show the dependence between the statements, $P(1|3)$ can be compared with $P(1)$. If the two probabilities are not equal, then the two events are dependent:

$$P(1|3) = \frac{P(13)}{P(3)} = \frac{0.063726}{0.25} = 0.254904$$

Consequently, as stated above, the truth of the two statements are dependent. Likewise

$$P(1|2) = \frac{P(12)}{P(2)} = \frac{0.063726}{0.25} = 0.254904$$

shows that the truth of statements 1 and 2 are dependent.

Finally, $P(123) = 0.016569 \neq P(1) \cdot P(2) \cdot P(3) = 1/64 = 0.015625$; therefore the truth of the three statements in the group are dependent.

PROBABILISTIC ANALYSIS OF RISK (PAR)

Many of the problems of the real world are just the concern about the uncertainty of the outcome of a venture. All sorts of examples can be conceived:

1. A piece of equipment fails and a number of undesirable consequences result.

2. A new product fails to win public acceptance.

3. The stock market goes in the unexpected direction.

The uncertainty would be removed if the truth or falsity of correctly phrased statements were known. For example, the truth or falsity of the statement "The stock market rose in the past three months" would completely remove the uncertainty about the direction of the stock market during the past three months. It is obviously impossible to establish with certainty the truth or falsity of such statements about the future, but there might be some ways of estimating the probability of the truth or falsity. The prudent man will make a "good" decision if he has a valid estimate of the probability of success. By "good" is meant that the measure of his success is greater than the measure of his failure.

Sometimes the group of analysts will have a store of information about the situation of interest, and all of this information can be used to produce the answer of greatest validity about the probability of the truth of the final statement. All of this information, conceivably, can be expressed in the form of declarative statements, each of which is either true or false. The truth or falsity of any statement may not be known with certainty, but the probability of the truth of a statement may be estimated. A natural way for the analysts to organize all of their information would be to order the statements that contain all of the information in such a way that the probability of the truth of any statement is conditional only upon the truth of preceding statements. A trivial example may clarify the structure of the analysis. Statement 3, "The automobile in which Mr. A is a passenger has a serious accident during the next 10 minutes," is of primary concern. There may be many other statements that affect the probability of the truth of this statement. For example, statement 1, "The automobile in which Mr. A is a passenger is in motion on a public thoroughfare." If the probability of the truth of statement 1 is low, then the probability of the truth of statement 3 is also low. The automobile is less likely to have an accident if it is standing still. The analysts might like to consider the truth of statement 2, "The automobile in which Mr. A is a passenger has an average speed of 80 mph during the next 10 minutes." The probability of the truth of statement 3 is larger, presumably, if the probability of truth of statement 2 is large. The analysts might express all of their information in probability statements such as the following:*

* The notation "1" means statement 1 is true, and "$\bar{1}$" means statement 1 is false. Similar notation applies to the other statements.

$P(1) = 0.4$

$P(2|1) = 0.2$

$P(2|\bar{1}) = 0.0$

$P(3|21) = 0.5$

$P(3|2\bar{1}) = 0.0$

$P(3|\bar{2}1) = 0.2$

$P(3|\bar{2}\bar{1}) = 0.0$

In instances where the conditional probabilities are zero, the events are considered impossible. For example, $P(2|\bar{1}) = 0.0$ implies that the average speed cannot be 80 mph when the automobile is not in motion.

Another example may provide additional clarification because of different phrasing. "An airplane has three motors and can fly as long as two are operational. Each of the three motors may fail because of faults inherent within the individual motors. Additionally, all three motors may fail simultaneously from a common cause such as exhaustion of the fuel. What is the probability that the plane will complete a mission from New York to Los Angeles without failure?" The analysts may formulate the problem as follows:

1. The common mode event* occurs
2. Engine No. 1 fails.
3. Engine No. 2 fails.
4. Engine No. 3 fails.
5. The plane fails.

$P(1) = 0.15$

$P(2|1) = 1.0$

$P(2|\bar{1}) = 0.4$

$P(3|1) = 1.0$

---

*A common mode failure is the occurrence of a single event that produces all of another set of failure events.

$P(3|\bar{1}) = 0.3$

$P(4|1) = 1.0$

$P(4|\bar{1}) = 0.2$

$P(5|\bar{2}34) = 1.0$

$P(5|2\bar{3}4) = 1.0$

$P(5|23\bar{4}) = 1.0$

$P(5|234) = 1.0$

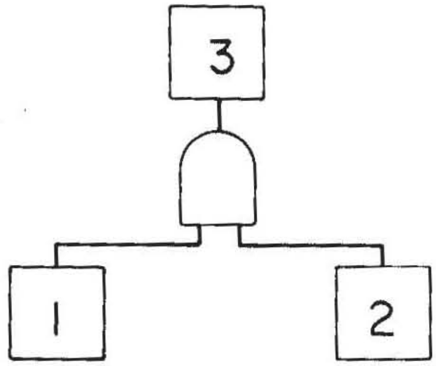
If statement 1 is true, then statements 2, 3, and 4 are true with probability of 1.0. If any two or more of statements 2, 3, and 4 are true, then statement 5 is true with probability of 1.0. Although there are $2^5 = 32$ possible outcomes or combinations of true and false conditions for five statements, for the conceptual experiment of the plane flight, it can be demonstrated that the eleven values listed above are sufficient to specify the probability of any event that can be formed as a subset of the 32 outcomes.

The analysis of this situation does not specify any order for the events that might occur. For example, the plane might fail from failure of any two or more engines in any order, or the common mode failure event might occur either before or after a single motor failed. The analysis contains several possible chronological sequences of events, and the single analysis includes all of them. The common mode failure event was included very naturally and produced no unusual complication in the analysis.

Fault Trees

The subject of fault trees is treated in some detail because it is currently a popular type of analysis. This treatment has the objective of showing that fault trees are merely special cases of the PAR method of analysis.

Fault trees consist of chains of events linked together by logical "and" and "or" gates. The "and" gate is represented by the following symbolism:



which means that event 3 occurs when both 1 and 2 occur. The generalized specification for a set of three declarative statements in the PAR analysis now becomes

$P(1) = a$

$P(2|1) = b$

$P(2|\bar{1}) = b$

$P(3|21) = 1$

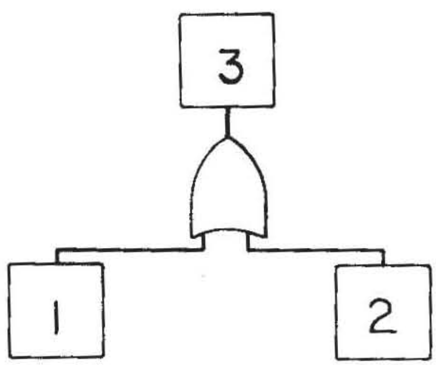
$P(3|2\bar{1}) = 0$

$P(3|\bar{2}1) = 0$

$P(3|\bar{2}\bar{1}) = 0$

Thus, rather than seven different values for this set of probabilities, there are only two, a and b, that are other than unity or zero.

Likewise, the "or" gate is represented by the following symbolism:



which means that event 3 occurs when either or both 1 and 2 occur. The generalized specification for a set of three declarative statements in the PAR analysis now becomes

$$P(1) = a$$

$$P(2|1) = b$$

$$P(2|\bar{1}) = b$$

$$P(3|21) = 1$$

$$P(3|2\bar{1}) = 1$$

$$P(3|\bar{2}1) = 1$$

$$P(3|\bar{2}\bar{1}) = 0$$

Again there are only two probabilities, a and b, that are different from zero or unity rather than seven.

When the number of events connected to an "or" gate become large, the number of entries that equal unity also becomes large. Actually, when k is the number of events, the number of entries is 2^k-1, which may be inconveniently large. This problem of a large number of entries can be avoided if the top event is redefined as "not-or" or the negation of the otherwise declarative statement. With this redefinition, the only data entry needed is

$$P(\bar{k}_n|\bar{k}_{n-1}, \bar{k}_{n-2} \ldots \bar{k}_1) = 0$$

In this fashion, either the "and" or the "or" gate requires only one data entry.

Every chain of "and" and "or" gates can be strung together to form the required fault tree, and the PAR analysis will produce the probabilities of all the constituent events. The actual computer program for the PAR analysis does not require any of the zero probabilities as input.

Common Mode Failures

Common mode failures may be important in a reliability analysis but appear to create computational difficulties when included in the usual analyses. A common mode failure occurs when the occurrence of a specified event results in the occurrence of each of several other events. For example, a power failure would cause the failure of all amplifiers on that power line.

The example discussed earlier of the airplane with three motors contained a common mode failure. The exhaustion of fuel caused each of the three motors to fail because of a common external cause. This and similar problems cause no special problems in the PAR analysis because the input is readily specified. The ease with which the PAR analysis handles this problem may result from the fact that all chronological sequences are handled simultaneously without explicit specification.

When the data listed earlier for the "airplane" problem were used as input to the PAR computer program, the results were:

P(1) = 0.15

P(2) = 0.490

P(3) = 0.405

P(4) = 0.320

P(5) = 0.3302

The question asked was, "What is the probability that this given airplane would successfully complete the trip from New York to Los Angeles?" The answer for the probability of failure for the specified input is P(5) = 0.3302. The common mode failure caused no special difficulty in the analysis because the input contained no features of a special nature.

Decision Trees

The PAR method of analysis and its computer program can be used to calculate the merits of alternative decisions that are the choice of an individual. This feature is possible because a value designated "RUE" (relative undesirability of the event) is assigned to each outcome of a probabilistic experiment. The product of the RUE and the probability of the event is a measure of the risk of that event. When a person has alternative choices about his future action he can evaluate the risks by the PAR analysis. The person will prudently choose the decisions that produce the smallest risk. A simple example will illustrate the type of analysis possible with PAR. A person is offered the choice of playing a game in which a coin is to be tossed. If the coin falls "head, the player receives 5 units. However, if the coin falls "tail," the player must pay 7 units. This much of the game is a poor one from the viewpoint of the player because on the average he will lose one unit for each play of the game. However, if the player loses the 7 units because the coin falls "tail," he is offered the choice of playing again with a new set of pay-offs. This second time, the fall of a "head" produces a gain of 10 units and the fall of a "tail" produces a loss of 3 units. What sort of decision should the player make? The player has three choices:

1. Don't play first time.

2. Play first time and quit.

3. Play first time and if he loses play second time.

When the necessary input is provided to the PAR computer program, the expected risks are as follows:

$E(R_1) = 0.000$

$E(R_2) = 1.00$

$E(R_3) = -0.75$

These values show what are fair and favorable wagers for each choice. Decision 3 is most favorable and indicates that a fair wager for the game is 0.75 unit. Any wager less than 0.75 unit is favorable, and larger wagers are unfavorable. If the player were forced to make decision 2, he should expect to be paid at least 1 unit to play or otherwise he can expect to come out a loser.

REFERENCES

1. J. W. Croach and L. M. Arnett. *Quantitative Safety Analysis*. USAEC Report DP-1207, E. I. du Pont de Nemours and Company, Savannah River Laboratory, Aiken, SC (April 1970).

2. A. Papoulis. *Probability, Random Variable, and Stochastic Processes*. McGraw-Hill, New York (1965).