DP-1255

AEC RESEARCH AND DEVELOPMENT REPORT

GRAVITY FLOW OF A STRATIFIED FLUID TO A SINK

J. W. CROACH

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Savannah River Laboratory

Aiken, South Carolina

Printed in the United States of America
Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.95

663234

DP-1255

Engineering and Equipment
(TID-4500, UC-38)

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by

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March 1971

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AIKEN, S. C. 29801

CONTRACT AT(07-2)-1 WITH THE
UNITED STATES ATOMIC ENERGY COMMISSION

ABSTRACT

Dual stream functions were adapted to a description of stratified flow, and a general vector equation for steady-state flow of a stratified liquid under gravity was developed. The stratified flow was required to be derivable from a static source fluid. A critical distribution of flow as a function of depth at the source was obtained by a variational method based on an extremal hypothesis that the flow is confined to a minimum layer thickness. Solutions were obtained for both channel and axisymmetric flow for the case where the source fluid is linearly stratified. Several sink geometries were considered for axisymmetric flow, and typical streamline solutions are given. Comparison of theoretical results with published experiments for channel flow and axisymmetric flow revealed generally satisfactory agreement.

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INTRODUCTION

Selective withdrawal of a fluid of a desired density from a stably stratified source in a reservoir is a problem of considerable interest. Stratification may arise from temperature variations or concentration gradients of dissolved salts. Of greatest practical importance is the relationship between the draw-off rate and the density of the withdrawn fluid when the density distribution of the stably stratified source is known and the draw-off structure is specified.

Early attention to this problem was reported by Craya[1] who considered selective withdrawal of one layer of a two-layer system; his predictions were later compared with experimental results obtained by Gariel.[2] Craya also developed a theory of a "Critical Regime" in stratified flow with hypotheses based on a condition of "minimum thrust" and "minimum energy flux."[3]

Yih[4,5] showed that inertial effects of density variations in a fluid could be conveniently handled by defining an associated flow of constant density through the velocity-density transformation $\sqrt{\rho_0}(u',v',w') = \sqrt{\rho}(u,v,w)$. Yih[5] also solved the problem of two-dimensional flow to a line sink for an inviscid fluid that was presumed to have a uniform density gradient at the source. Yih showed that fluid separation would not be possible when the Froude number in the channel exceeds $\frac{1}{\pi}$.

Debler[6,5] conducted experiments with saltwater stratified to form a constant density gradient in a rectangular tank with flow to a line sink. These experiments led Debler to propose that the height of the flow region below a dividing streamline is characterized by a modified Froude number of $\frac{1}{\pi}$, but no theoretical basis was given.

Kao[7,5] developed an analysis for a two-dimensional flow with a velocity discontinuity between an underlying stream of moving fluid and an overlying stagnant region.

Koh[8,5] through a perturbation method analyzed two-dimensional stratified flow with viscous effects when the Reynolds number is not large. Experimental data were obtained with both salt solutions and thermally stratified water. Velocity distributions were obtained and correlated well with the theoretical analysis.

Harleman, Morgan, and Purple[9] performed experiments with axisymmetric flow to a sink in a large tank containing two layers. Results were interpreted in terms of a critical Froude number for discharge of the lower layer such that the upper layer remains stagnant.

Wood[10] reported analyses and experiments for selective withdrawal of stratified layers from a reservoir through a constriction.

The objective of the work reported here is primarily to consider theoretically the steady-state gravity flow of a continuously stratified, inviscid fluid to a sink subject to two conditions:

- The pressure and velocity of the moving stream are required to be continuous at the boundary with overlying or underlying stagnant regions.
- The moving stream is confined to the minimum possible layer thickness consistent with reasonable physical constraints.

SUMMARY

Dual stream functions were adapted to a description of stratified flow; ψ is associated with layers of constant density, and X represents an appropriate orthogonal set of surfaces. A general equation for steady-state flow of a stratified liquid under gravity was developed by vector methods with the dual stream functions. With appropriate selection of the X function for two-dimensional horizontal flow and for axisymmetric flow, the general equation was shown to reduce to the second order partial differential equations reported by Yih[5] for a single stream function.

The governing steady-state equation of flow contains two arbitrary functions. One of these functions is determined if the static source fluid is specified (density versus depth) together with a designation of a "discharge elevation," which may be considered as an elevation of maximum potential energy in the source fluid. The second arbitrary function is determined if it is required that the flow be confined to a minimum layer thickness about the discharge elevation. The latter extremal condition yields a classical problem of Lagrange from the calculus of variations. The solution of the variational problem leads to critical flow equations that uniquely determine the distribution of flow as a function of density. For a source fluid with linear variation of density with depth, it is shown that equal flow is obtained from equal density intervals (or equal depth intervals) in the source; the equations for two-dimensional horizontal flow (channel flow) are analytically solved to yield density and velocity as a function of depth in the channel. In this case, it is shown that the velocity varies as the cosine-squared function of the depth in the channel; the flow is characterized by a modified Froude number of $\frac{1}{\pi}$. For any source distribution in

critical channel flow, Richardson's number may not be less than $\frac{1}{2}$.

In the development of the general stratified flow by means of dual stream functions and in the derivation of the critical flow conditions from the extremal hypothesis, it was not necessary to assume any restrictions on the magnitude of density variation in the fluid.

Solutions of the governing partial differential equations for steady gravity flow were obtained for both channel and axisymmetric flow; the source fluid was assumed to be linearly stratified. In these solutions, which were obtained by iterative computation of the finite difference equations, a satisfactory algorithm was found to represent the requirement that the fluid boundary bordering a stagnant region must have a zero velocity. This condition is a form of free boundary that leads to a Cauchy or third boundary problem. A consequence of this boundary condition is that a region of constant density fluid (isopycnic wedge) must be postulated to exist between the flowing fluid and any contiguous region of stably stratified stagnant fluid.

The solution for channel flow was for a line sink at the base of a vertical dam terminating the channel. Total flow in a channel, per unit width, is determined by the critical flow conditions and is known prior to the solution of the governing partial differential equation. Additional information obtained from the solution of the partial differential equation is the identification of streamlines and the capability to calculate velocity and pressure.

Several sink geometries were considered for axisymmetric flow, and typical streamline solutions are given. The total flow for axisymmetric geometries was calculated in terms of the equivalent width of a horizontal channel that would yield the same total flow from the given source and stream thickness. For instance, in the important case of flow to a point sink where the steady flow is derived from a source height h in a linearly stratified fluid, the total flow is the same as that which can be supported by horizontal channel flow from the same source and a channel width of 0.63 h. The numerical value 0.63 is determined as an eigenvalue of the partial differential equation that meets specified boundary requirements.

In addition to the point sink in axisymmetric flow, solutions were found for circular curvilinear sinks and open holes of various radii. The case of a point sink beneath a rigid disk is also considered.

In both the channel and axisymmetric cases, it was not possible to find steady-state solutions meeting the zero velocity boundary requirement that spanned the entire distance from sink to source. The implication is that a transition region of unsteady flow must exist between the entrance of the flowing stream to the steady-state region and the stagnant source fluid (if it exists) even farther upstream.

Comparison of the theoretical results with Debler's[6] experiments for channel flow and with the experiments of Harleman, Morgan, and Purple[9] for axisymmetric flow revealed generally satisfactory agreement. The theory supports Debler's proposal that selective withdrawal in two-dimensional channel flow is characterized by a Froude number of $\frac{1}{\pi}$.

THEORY

BASIC EQUATIONS OF STEADY FLOW

General

The fluid is assumed to be incompressible, inviscid, and nondiffusive. Effects of wind, tide, and Coriolis force are neglected. Only those stratified flows are considered that are derivable from a stagnant, stably stratified source. The latter assumption is intended to imply that the Bernoulli condition holds along streamlines and that, if the velocity of the fluid is set to zero, the distribution of pressure and density with height will correspond to that of a stably stratified fluid; i.e., the density increases monotonically with depth.

While only steady flows are considered in this report, these must be regarded as limiting conditions that describe slowly changing unsteady flows. Interest is focused on the conditions of flow into a sink of some sort, with particular emphasis on selective withdrawal of the fluid. In the vicinity of a sink, the conditions for selective withdrawal may be developed by steady-state considerations, even though the character of the flow remote from the sink is time-dependent.

Associated Flow

Let $\vec{q}\,'$ be the velocity vector for the stratified flow. As Yih showed,[4,5] an associated flow may be defined by the transformation,

$$\vec{q} = \sqrt{\frac{\rho}{\rho_o}}\ \vec{q}\,' \qquad (1)$$

where $\vec{q}$ is the velocity vector of the associated flow of constant reference density ρ_o, and ρ is the variable density of the stratified flow. This elegant transformation accounts for the inertial effects of the variable density, and will therefore be used throughout the analysis. (The prime is applied to the actual flow instead of the associated flow in the definition to avoid repeated use of the prime.) If density variations are small, it is unnecessary to transform solutions for the associated flow back to the actual flow with variable density. The constant reference density, ρ_o, is arbitrary, but a convenient assignment will be made when the source fluid is related to the elevation of discharge at the sink.

Representation by Dual Stream Functions

Stratified flow, as shown by Yih,[5] may be characterized as layerwise irrotational, if it is derived from a rest condition at the source. The vorticity vector is normal to the density gradient. Essentially any flow field of an incompressible fluid may be represented by expressing the velocity vector $\vec{q}$ as the vector product of two scalar gradients $\nabla\psi$ and $\nabla\chi$.[11] The method is well suited to description of stratified flow; ψ may be associated with surfaces of constant density in the flow (ψ is a function of ρ); χ represents surfaces (χ = constant) orthogonal to the isopycnic surfaces. Thus,

$$\vec{q} = \nabla\chi \times \nabla\psi$$
$$\text{and} \quad \nabla\chi \cdot \nabla\psi = \nabla\chi \cdot \nabla\rho = 0 \qquad (2)$$

($\nabla\chi$ need not have been chosen orthogonal to $\nabla\psi$, but it is convenient to do so.) The interpretation of ψ and χ as stream functions is in the usual way. In Figure 1, $\psi_a - \psi_b$ is a measure of the quantity of flowing fluid bounded by surfaces $\psi = \psi_a$ and $\psi = \psi_b$; similarly, the quantity of fluid flowing between $\chi = \chi_a$ and $\chi = \chi_b$ is $\chi_a - \chi_b$. In the stagnant source region, the surfaces

ψ = constant, since they coincide with layers of constant density, must be horizontal planes while the orthogonal surfaces χ = constant must be vertical planes.

If the flow exhibits selective withdrawal of a stratified fluid, with the flow exiting to a sink, then an upper bounding layer $\psi(\rho_u)$ or a lower boundary layer $\psi(\rho_\ell)$ (or both) exists. Fluid above and below the respective upper and lower boundaries is assumed to be stagnant. Conditions of hydrostatic equilibrium then require the bounding surfaces to be horizontal planes, and the velocities at these boundaries must be zero.

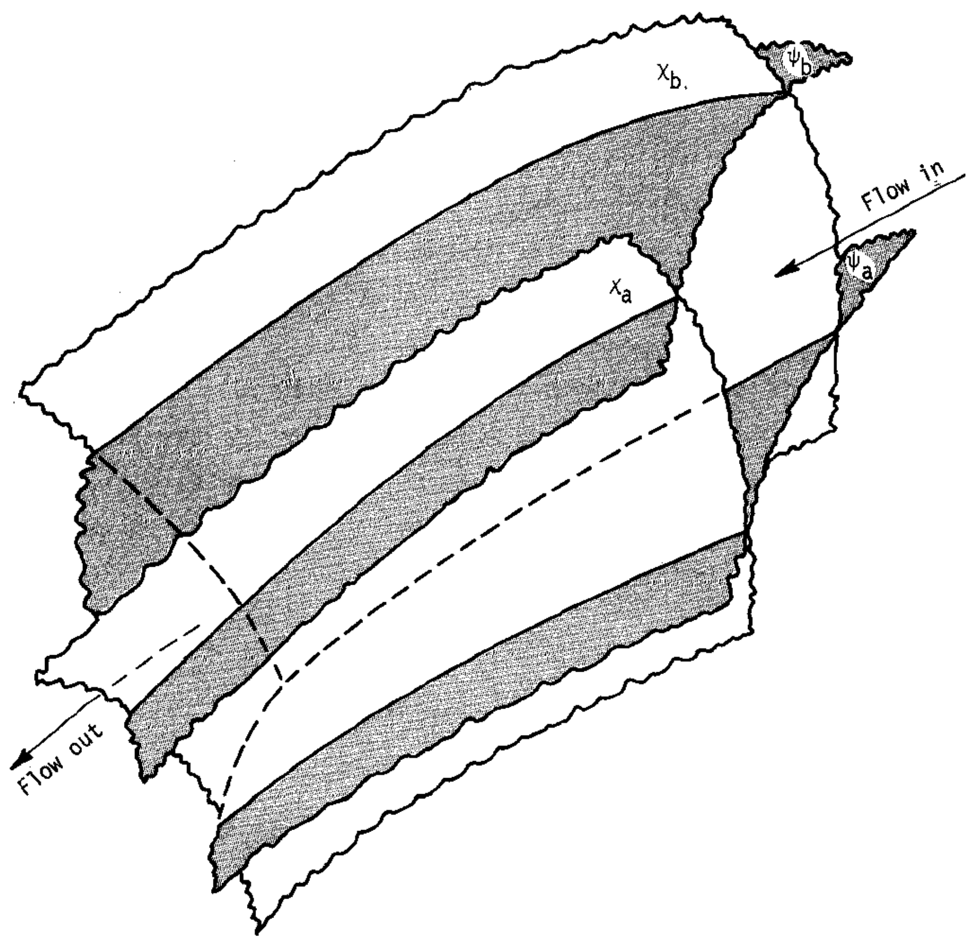


FIG. 1 FLOW DEFINED BY DUAL STREAM FUNCTIONS ψ AND χ

General Equation of Stratified Flow

The Euler equation of fluid motion for the associated flow in a gravitational field (reference density ρ_o) is

$$\rho_o \vec{a} + g\rho \nabla z + \nabla P = 0 \tag{3}$$

where $\vec{a}$ is the acceleration vector, P is the pressure, and g is the gravitational constant; z represents the elevation of the fluid above a reference plane. Lagrange's form of the acceleration is

$$\vec{a} = \frac{\partial \vec{q}}{\partial t} + \nabla\left(\frac{q^2}{2}\right) + \vec{\zeta} \times \vec{q}$$

$\vec{\zeta}$ is the vorticity vector and q is the scalar speed. In steady state, $\frac{\partial \vec{q}}{\partial t} = 0$; with equation (3),

$$\rho_o \nabla\left(\frac{q^2}{2}\right) + g\rho \nabla z + \nabla P = \rho_o \vec{q} \times \vec{\zeta} \tag{4}$$

The term $\vec{q} \times \vec{\zeta}$ combined with equation (2) yields

$$\vec{q} \times \vec{\zeta} = (\nabla\chi \times \nabla\psi) \times \vec{\zeta}$$

$$= (\nabla\chi \cdot \vec{\zeta})\nabla\psi - (\nabla\psi \cdot \vec{\zeta})\nabla\chi$$

Since the vorticity vector is orthogonal to the density gradient (and therefore to $\nabla\psi$), $\nabla\psi \cdot \vec{\zeta} = 0$. Thus, $\vec{q} \times \vec{\zeta} = (\nabla\chi \cdot \vec{\zeta})\nabla\psi$. By the definition of vorticity,[11] $\vec{\zeta} = \nabla \times \vec{q}$. Equation (4) may now be written as

$$\rho_o \nabla\left(\frac{q^2}{2}\right) + g\rho \nabla z + \nabla P = \rho_o G \nabla\psi \tag{5}$$

where G is a scalar quantity given by

$$G = \nabla\chi \cdot \nabla \times (\nabla\chi \times \nabla\psi) \tag{6}$$

The Bernoulli quantity is defined as

$$H = \rho_o \frac{q^2}{2} + g\rho z + P \qquad (7)$$

Equations (5) and (7) combine to yield

$$\nabla H - gz\nabla\rho = \rho_o G\nabla\psi \qquad (8)$$

$\nabla\rho$ and $\nabla\psi$ are in the same direction; the derivative in the direction of the density gradient gives

$$\rho_o G = \frac{dH}{d\psi} - gz\,\frac{d\rho}{d\psi} \qquad (9)$$

Equation (9) and the method of derivation parallel Yih's derivation for two-dimensional and axisymmetric flow.[5] In general, the expression for G is formidable, but is tractable for the channel flow (two-dimensional) and axisymmetric cases.

Channel Flow (Two rectangular dimensions)

An appropriate χ function is simply $\chi = y$, where y is the cross channel dimension. Let the dimension lengthwise of the channel be x and let $\hat{i}$, $\hat{j}$, and $\hat{k}$ be unit vectors in the x, y, and z directions. Thus $\nabla\chi = \hat{j}$. Since $\nabla\chi \cdot \nabla\psi = 0$, $\hat{j} \cdot \nabla\psi = 0$ and the $\hat{j}$ component of $\nabla\psi$ is zero. The quantity G becomes

$$\begin{aligned} G &= \hat{j} \cdot \nabla \times (\hat{j} \times \nabla\psi) \\ &= \hat{j} \cdot [(\nabla \cdot \nabla\psi)\hat{j} - (\nabla \cdot \hat{j})\nabla\psi] \\ &= \nabla \cdot \nabla\psi = \nabla^2\psi \end{aligned}$$

and, the channel equation from equation (9) is

$$\rho_o \nabla^2\psi = \frac{dH}{d\psi} - gz\,\frac{d\rho}{d\psi} \qquad (10)$$

Equation (10) is the same as Yih's equation,[5] and ψ is the usual Lagrangian stream function. Thus, the horizontal and vertical velocity components are $q_x = -\frac{\partial\psi}{\partial z}$ and $q_z = \frac{\partial\psi}{\partial z}$.*

Axisymmetric Flow

The X stream function assumes the form $X = b\theta$, where θ is the azimuthal angle and b is a constant. Let r be the radius and z the vertical dimension; unit vectors $\hat{\theta}$, $\hat{r}$, and $\hat{k}$ are in the direction of θ, r, and z. $\nabla X = \frac{b\hat{\theta}}{r}$, and ψ does not depend upon θ.

The G quantity is

$$G = \frac{b\hat{\theta}}{r} \cdot \nabla \times \left(\frac{b\hat{\theta}}{r} \times \nabla\psi\right)$$

The reduction of G is straightforward, and the result is

$$G = \frac{b^2}{r^2} \left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$

Thus, the axisymmetric equation is

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{r^2}{\rho_o b^2}\left(\frac{dH}{d\psi} - gz\frac{d\rho}{d\psi}\right) \qquad (11)$$

Equation (11) is the same as Yih's[5] except that it contains the constant "b." The difference arises from Yih's choice of ψ as the usual form for Stokes stream function:

$q_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}$ (radial velocity component)*

$q_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$ (vertical velocity component)*

* The choice of signs is to reflect a fluid velocity directed toward the origin (negative) when ψ increases with z.

Equation (11) implies

$$q_r = -\frac{b}{r}\frac{\partial\psi}{\partial z}$$

$$q_z = \frac{b}{r}\frac{\partial\psi}{\partial r}$$

Inclusion of the constant b is convenient as an explicit parameter (which could otherwise be incorporated into ψ). If b is considered to have the dimension of length, the dimensions of ψ are the same in both equations (10) and (11), thus avoiding the inconvenience caused by the fact that the Lagrange and Stokes stream functions have different dimensions.

CRITICAL HYPOTHESIS

General

Equations (9), (10), and (11) for stratified flow contain terms H and ρ that may be regarded as functions of ψ. It is desired to find suitable expressions for H and ρ (actually $\frac{dH}{d\psi}$ and $\frac{d\rho}{d\psi}$) in terms of ψ, based on the assumption that the flow originates from a stably stratified rest condition.

Let the stably stratified source be characterized by the function $Z(\rho)$, where Z is the elevation of fluid with density ρ above a zero reference plane. The reference plane is chosen at the elevation where the density is ρ_o, the reference used in the derivation of the equations of flow. The Bernoulli quantity H is given by equation (7). At the source, because q is zero and the elevation z for density ρ is $Z(\rho)$,

$$H = g\rho Z + P \tag{12}$$

Differentiate (12) with respect to ρ, and note that:

$$\frac{dP}{d\rho} = \frac{dP}{dZ}\frac{dZ}{d\rho} = -g\rho\frac{dZ}{d\rho} \tag{13}$$

gives $\frac{dH}{d\rho} = gZ$ (14)

When $\rho = \rho_o$, $Z(\rho_o) = 0$ and $\frac{dH}{d\rho} = 0$; thus H has an extremal at the reference plane; the extremal is clearly a maximum, since $\frac{d^2H}{d\rho^2} = g\,\frac{dZ}{d\rho}$ and $\frac{dZ}{d\rho}$ must be negative.

While a given stagnant source distribution $Z(\rho)$ is physically invariant with respect to the choice of a reference elevation, the above result shows that the same is not true of $H(\rho)$. The function H is uniquely specified only when ρ_o [for which $Z(\rho_o) = 0$] is specified. This implies a definite relationship between the elevation of fluid discharge at the sink and the reference elevation corresponding to ρ_o at the source. The effective elevation of discharge may be taken, by definition, as that elevation in the source fluid with density ρ_o. Often the elevation of discharge is apparent, such as the case where a point sink is located in a large body of fluid remote from any rigid boundaries or the fluid surface. The same is true for a horizontal line sink at the base of a dam terminating a channel. In more complex exit structures, the effective discharge elevation may not be known explicitly until the flow problem is solved.

A property of the streamline with density ρ_o is that it has a maximum energy, H, with respect to adjacent streamlines, as was indicated by equation (14). If the streamline ρ_o is at the same elevation near discharge that it had in the source region, then the streamline ρ_o will have a maximum velocity with respect to its neighbors. The coincidence of maximum velocity and maximum kinetic energy is true for the associated flow as a result of the transformation given by equation (1). In the actual flow, streamlines of maximum kinetic energy and maximum velocity do not coincide, although they are close together if the density gradient is small.

By equation (14),

$$\frac{dH}{d\psi} = gZ \frac{d\rho}{d\psi} \tag{15}$$

Since Z is known (given the source and the reference or discharge elevation), the quantity $\frac{dH}{d\psi}$ is determined, if $\frac{d\rho}{d\psi}$ can be found.

The Extremal Hypothesis

The stream function ψ may be regarded as a mapping function on the source distribution: $\psi = \psi(\rho)$. Consider a given source fluid in some specified flow geometry where a given steady flow is being withdrawn to a specified sink configuration. If $\psi_1 = \psi(\rho_1)$ and $\psi_2 = \psi(\rho_2)$ are two arbitrary stream surfaces with corresponding layer densities of ρ_1 and ρ_2, then the extremal hypothesis is that:

- $\delta \int_{\rho_1}^{\rho_2} d\psi = 0$
- The Bernoulli equation (7) is satisfied.

(16)

Physically, the hypothesis may be interpreted as follows: Out of the set of all possible mapping functions $\psi(\rho)$ that satisfy the Bernoulli condition and that yield the specified total flow to the sink, the flow will tend to a unique function, $\psi_m(\rho)$, that satisfies the extremal condition of equation (16). The permitted variations of $\psi(\rho)$ between ρ_1 and ρ_2 are only those functions for which the dependent variables, z, q, and P have fixed, specified values at ρ_1 and ρ_2. The physical model and the solutions that are obtained by application of (16) require the extremal condition to be a maximum: the flow between ρ_1 and ρ_2 will be the maximum possible consistent with the specified conditions. A consequence of the hypothesis is that the flow will tend to confine itself to the least possible layer thickness, subject to the specified conditions: the quantity $Z(\rho_u) - Z(\rho_\ell)$ will be a minimum, where

ρ_u and ρ_ℓ are respectively the densities of the upper and lower limits of withdrawal. This follows from equation (16), if ρ_1 and ρ_2 are identified with the boundary layers ρ_u and ρ_ℓ; an extremal (maximum) flow between stream surfaces ρ_u and ρ_ℓ implies that these surfaces have the least possible separation that will accommodate that flow.

The extremal of equation (16) is believed to be an appropriate critical condition that applies everywhere in the flow. Let the function that satisfies (16) be $\psi_m(\rho)$. Consider any stream tube extending from the source to the sink. The Bernoulli condition holds everywhere along the tube. If $\psi_m(\rho)$ can be found at any cross section along the tube, then this solution must apply over the length of the tube since the flow distribution as a function of density does not involve velocity, elevation, pressure, or cross-sectional area of the tube. It would be very difficult to solve the general variational problem that is posed by (16) for an arbitrary cross section of the flow, because q, z, P must be treated as dependent variables in the variation; in addition, several other dependent variables are required to specify the geometry of a surface normal to the flow. If it is accepted that (16) does apply and is independent of the geometry, then the desired mapping function $\psi_m(\rho)$ is uniquely specified once the bounding layers are known. It is therefore possible to devise a hypothetical geometry of flow that has the appropriate boundaries in the source region and that greatly simplifies the solution of the variational problem. The simplest possible geometry is one in which flow from the source, with bounding layers ρ_u and ρ_ℓ, is ultimately constrained to flow horizontally in a channel with vertical parallel walls (again with bounding layers ρ_u and ρ_ℓ).

Critical Equations for Horizontal Channel Flow

In this development, the density ρ of the source is generally considered to be the independent variable. Let ρ_o be the reference density, which is taken as that of the discharge elevation in the source; ρ_o is that density (streamline) that has maximum velocity in the channel. $Z(\rho)$ is the height of density ρ in the source with $Z(\rho_o) = 0$. Consider a horizontal flow in the channel (which might physically occur in a region that is far from both the channel entrance and the exit structure that defines the sink). Let $z(\rho)$ be the height of density ρ in the channel, and $q(\rho)$ be the velocity (vertical component zero). The Bernoulli condition for such a flow is

$$H(\rho) = \rho_o \frac{q^2}{2} + g\rho z + P \tag{17}$$

Differentiate (17) with respect to ρ,

$$\dot{H} = \rho_o q\dot{q} + gz + g\rho\dot{z} + \dot{P} \tag{18}$$

(Here and subsequently, the superior dot denotes a derivative with respect to ρ, e.g., $\dot{H} = \frac{dH}{d\rho}$.) In the channel with horizontal flow and zero acceleration, the pressure is the same as the hydrostatic pressure corresponding to the density distribution with depth in the channel; i.e.,

$$\frac{dP}{dz} = -g\rho \qquad \text{or} \qquad \dot{P} = \frac{dP}{d\rho} = \frac{dP}{dz}\frac{dz}{d\rho} = -g\rho\dot{z} \tag{19}$$

Substituting (14) and (19) into (18) yields

$$\rho_o q\dot{q} = g(Z-z) \tag{20}$$

In the channel, the streamline with maximum velocity implies $\dot{q} = 0$. This streamline of maximum velocity is the one that corresponds to the discharge elevation $Z = 0$ and therefore to density ρ_o. From equation (20), $z(\rho_o) = Z(\rho_o) = 0$. Thus, the maximum velocity in the channel is at the density ρ_o, which has the same elevation both in the source and in the channel. This elevation serves as the reference zero for both z and Z. (Physically, one may visualize a sink at elevation $z = 0$ somewhere far downstream.)

Now consider the flow per unit channel width between streamlines ρ and $\rho + d\rho$ (between elevations z and $z + dz$):

$$d\psi = q\,dz$$

The extremal condition (16) as applied to the channel flow may be written as

$$\delta \int_{\rho_1}^{\rho_2} d\psi = \delta \int_{\rho_1}^{\rho_2} q\,dz = \delta \int_{\rho_1}^{\rho_2} q\dot{z}\,d\rho = 0 \qquad (21)$$

Equation (21) is subject to the Bernoulli condition. But in the horizontal channel flow, the derivative of the Bernoulli equation led directly to (20), so that it may be used as an appropriate condition to impose on (21). The problem is now in the typical form of the classical problem of Lagrange from the calculus of variations. It may be solved by the Lagrange-multiplier method to obtain the Euler-Lagrange equations. The function,

$$F = q\dot{z} + \lambda(\rho_o q\dot{q} + gz - gZ) \qquad (22)$$

is formed in which λ is a Lagrange multiplier (a function of ρ) The Euler-Lagrange equations are:

$$\frac{\partial F}{\partial q} - \frac{d}{d\rho}\frac{\partial F}{\partial \dot{q}} = 0 \qquad\qquad \frac{\partial F}{\partial z} - \frac{d}{d\rho}\frac{\partial F}{\partial \dot{z}} = 0$$

These produce immediately:

$$\dot{z} + \lambda\rho_o q - \frac{d}{d\rho}(\rho_o \lambda q) = 0$$

$$q\lambda - \frac{d}{d\rho}\, q = 0$$

Elimination of λ from the two equations yields

$$\dot{z} = \frac{\rho_o}{g} q \frac{d^2q}{d\rho^2} = \frac{\rho_o}{g} q\ddot{q} \tag{23}$$

If equation (20) is differentiated with respect to ρ and $\dot{z}$ is eliminated via (23),

$$2q\ddot{q} + \dot{q}^2 = \frac{g}{\rho_o} \dot{Z} \tag{24}$$

Equation (24) is a second order nonlinear differential equation with q as the dependent variable and ρ as the independent variable. ($\dot{Z}$ is a known function of ρ, if the stratification of the source is known.)

A specific solution of (24) must invoke two boundary conditions. If free (fluid) boundaries ρ_u and ρ_ℓ are specified, then the two boundary conditions are $q(\rho_u) = q(\rho_\ell) = 0$ and the solution is then determined. In this case, the streamline of maximum velocity (ρ_o) is also determined; thus, ρ_o, ρ_u, and ρ_ℓ are not all independent.

It is convenient to define a flow region bounded by ρ_o and either ρ_u or ρ_ℓ. The boundary conditions for equation (24) then become $\dot{q}(\rho_o) = 0$ and either $q(\rho_u) = 0$ or $q(\rho_\ell) = 0$. To avoid repetitious discussions of boundary conditions, the flow will henceforth be treated with boundaries ρ_o at the elevation of discharge and $\rho_u = \rho_*$ as the upper boundary of flow. This corresponds to a physical model in which the reference plane $z = 0$ defines a frictionless bottom to the entire region of flow (including the source, and in which the elevation of discharge is similarly $z = 0$); i.e., the sink is located in the reference plane. Adaptation of problems and solutions to different boundary conditions will not be treated explicitly here. (Generally ρ_* could just as well represent a lower boundary as an upper boundary.)

If equations (23) and (24) are combined to eliminate the term in $q\ddot{q}$,

$$\dot{q}^2 = \frac{g}{\rho_o} (\dot{Z} - 2\dot{z}) \tag{25}$$

Once equation (24) is solved so that q is a known function of ρ, $z(\rho)$ may be found via equation (20); i.e.,

$$z = Z - \frac{\rho_o}{g} q\dot{q} \tag{26}$$

$z = Z$ at both boundaries ρ_o and ρ_*:

$$z(\rho_o) = Z(\rho_o) = 0$$

$$z(\rho_*) = Z(\rho_*) = h$$

h is the depth of the withdrawal layer at the source. The depth of the horizontal flow in the channel is also h.

It is possible to obtain an expression for $\ddot{\psi} = \frac{d^2\psi}{d\rho^2}$ as follows: For the horizontal channel flow being considered,

$$d\psi = q\,dz \qquad \text{or}$$

$$\dot{\psi} = q\dot{z} \tag{27}$$

With equations (25) and (27),

$$\dot{\psi} = \frac{q\dot{Z}}{2} - \frac{\rho_o}{2g} q\dot{q}^2$$

Differentiating with respect to ρ gives

$$\ddot{\psi} = \frac{q\ddot{Z}}{2} + \frac{\dot{q}\dot{Z}}{2} - \frac{\rho_o}{2g} \frac{d}{d\rho} (q\dot{q}^2)$$

But equation (24) may be written as

$$\frac{d}{d\rho} (q\dot{q}^2) = \frac{g}{\rho_o} \dot{q}\dot{Z}$$

Combining the last two equations yields

$$\ddot{\psi} = \frac{q}{2} \ddot{Z} \tag{28}$$

If the source stratification is linear [i.e., if $Z(\rho)$ is linear in ρ], then $\ddot{Z}$ is a constant and $\ddot{Z}$ is zero. For a linear source stratification, equation (28) shows that $\ddot{\psi}$ is zero, and $\dot{\psi}$ is a constant; i.e., $\frac{d\psi}{d\rho}$ (and therefore $\frac{d\rho}{d\psi}$) is a constant. Equal density intervals, within the boundaries of flow, yield equal flows.

In general, the stream function $\psi(\rho)$ may be interpreted as the amount of fluid flowing per unit channel width between streamline ρ and streamline ρ_o. Thus $\psi(\rho_o) = 0$. $\psi(\rho)$ may be determined by combining equations (23) and (27) to obtain

$$\dot{\psi} = \frac{\rho_o}{g} q^2 \ddot{q} \tag{29}$$

A remarkable feature of the critical equations is that no restriction is necessary on the density range to which they apply. They presumably apply regardless of the magnitude of the density variation, although most cases of practical significance are indeed restricted to relatively small variations in density.

Richardson's Number in Critical Channel Flow

Richardson's number is defined as[12]

$$R = -\frac{g}{\rho}\left(\frac{d\rho}{dz}\right)\Big/\left(\frac{dq'}{dz}\right)^2$$

In terms of the associated flow ($dq' = \sqrt{\frac{\rho_o}{\rho}}\, dq$), with a rearrangement of variables,

$$R = -\frac{g}{\rho_o}\left(\frac{dz}{d\rho}\right)\Big/\left(\frac{dq}{d\rho}\right)^2 = -\frac{g}{\rho_o}\frac{\dot{z}}{\dot{q}^2} \tag{30}$$

(Actually, dq' is not exactly equal to $\sqrt{\frac{\rho_o}{\rho}}\, dq$, but the difference is small if the density variation over the stratification is small.) In any case, it is suggested that (30) is the best generalization of Richardson's number in terms of the associated flow. If (30) is combined with (25),

$$\frac{\dot{Z}}{\dot{z}} = \frac{dZ}{dz} = 2 - \frac{1}{R} \qquad (31)$$

In (31), both $\dot{Z}$ and $\dot{z}$ must be negative in a stable steady flow; therefore, the minimum value of R permitted by (31) is $\frac{1}{2}$. This minimum value is approached as $\dot{z}$ approaches $-\infty$. Note that $\dot{z} = -\infty$ corresponds to $\frac{d\rho}{dz} = 0$; i.e., a zero density gradient. Apparently, Richardson's number must be $\geq \frac{1}{2}$ everywhere in a steady two-dimensional horizontal flow derived by gravity from a source fluid at rest.

Numerical Solutions of Critical Equations

Systematic solution of the critical channel equations may be obtained by Runge-Kutta methods. The following set of equations may be solved either simultaneously or sequentially in the indicated order:

$$2q\ddot{q} + \dot{q}^2 = \frac{\rho_o}{g}\dot{Z} \qquad (24)$$

$$z = Z - \frac{\rho_o}{g}q\dot{q} \qquad (26)$$

$$\dot{\psi} = \frac{\rho_o}{g}q^2\ddot{q} \qquad (29)$$

It is best to proceed as follows: Assume a maximum velocity $q_o = q(\rho_o)$ for the reference plane, $z = 0$. Integrate by steps until $q = 0$, thus establishing ρ_* at the boundary. In this way, the height of the flow for the assumed q_o is found as $h = z(\rho_*) = Z(\rho_*)$. The initial value $\psi(\rho_o) = 0$ permits integration of (29). The total flow per unit channel width is thus determined as $\psi_o = \psi(\rho_*)$. By obtaining a series of such solutions, ψ_o is established as a function of h.

Solutions for Constant Density Gradient at the Source

When the source fluid is stratified so that the density varies linearly with depth, straightforward analytical solutions are derivable for the horizontal channel equations. Let $\dot{Z} = -k$. The following relationships are developed:

$$\frac{\rho-\rho_*}{\rho_o-\rho_*} = \frac{2}{\pi}\left[\sin^{-1}\sqrt{\frac{q}{q_o}} - \sqrt{\frac{q}{q_o}\left(1-\frac{q}{q_o}\right)}\right] \tag{32}$$

$$q_o = \frac{2h}{\pi}\left(\frac{g}{\rho_o k}\right)^{\frac{1}{2}} \tag{33}$$

where q_o is the speed of fluid at reference plane $z(\rho_o) = 0$

$$h = k(\rho_o-\rho_*) \tag{34}$$

$$q = q_o \cos^2 \frac{\pi z}{2h} \tag{35}$$

The relation of (36) is to be interpreted that a streamline ρ of elevation $Z(\rho)$ in the source has an elevation $z(\rho)$ in the horizontal channel flow.

$$Z = z + \frac{h}{\pi}\sin\frac{\pi z}{h} \tag{36}$$

Other useful relationships are:

$$\dot{\psi} = q\dot{z} = -\frac{kq_o}{2} = -\frac{h}{\pi}\left(\frac{gk}{\rho_o}\right)^{\frac{1}{2}} \tag{37}$$

$$\psi(\rho) = \frac{h}{\pi}\left(\frac{gk}{\rho_o}\right)^{\frac{1}{2}}(\rho-\rho_*) \tag{38}$$

$$\psi_o = \frac{h^2}{\pi}\left(\frac{g}{\rho_o k}\right)^{\frac{1}{2}} \qquad \text{(total flow)} \tag{39}$$

$$h = \left(\frac{\pi^2\rho_o k\psi_o^{\,2}}{g}\right)^{\frac{1}{4}} \tag{40}$$

$$\frac{\psi}{\psi_o} = \frac{Z}{h} \tag{41}$$

As was predicted from equation (28), the constant density gradient at the source leads to a linear relationship of the flow ψ with ρ, as shown by (37) and (38).

Of particular interest in any stratified gravity flow exhibiting fluid separation (selective withdrawal) is the behavior of the fluid in the vicinity of the boundary of separation. Even for nonlinear source stratifications, the conditions close to this boundary must, in general, be well represented by the behavior of an appropriately selected linear distribution. Thus for small vertical displacements near the separating boundary, ρ_* at elevation $z = h$ in the horizontal channel flow, the flow-density distribution may be considered constant ($\dot{\psi} = -\varepsilon$). (If the source is linearly stratified, $\dot{\psi} = -\varepsilon$ over the entire depth of the channel flow.) Since $\dot{\psi} = q\dot{z}$, $\dot{z} = -\frac{\varepsilon}{q}$ in the vicinity of boundary ρ_*. As ρ approaches ρ_*, $q(\rho)$ approaches zero; therefore the magnitude of $\dot{z}$ must become infinite. Since $\dot{z}$ is the reciprocal of the density gradient, $\frac{d\rho}{dz}$, the density gradient approaches zero as ρ approaches ρ_*. From equation (25), $(\dot{q})^2$ must approach ∞ in the same way as $|\dot{z}|$ (since $\dot{Z}$ remains finite). The ratio of $|\dot{z}|$ to $(\dot{q})^2$ approaches a constant value as ρ nears ρ_*, so that the Richardson's number approaches an asymptotic value. The limiting value of Richardson's number is readily apparent from (31), which yields $R = \frac{1}{2}$ as $|\dot{z}|$ becomes infinite. Thus the phenomenon of fluid separation, for critical horizontal flow, is associated with the approach of Richardson's number to the limiting value $\frac{1}{2}$.

Alternative Critical Hypotheses

Craya[3] employed a variational approach in his analysis of stratified flow. He developed formulas for the "critical regime" corresponding to minimum thrust and to minimum energy flux. His two relationships did not yield the same results. Neither the minimum thrust nor the minimum energy flux hypothesis corresponds to the extremal hypothesis of this paper, which might be termed maximum flow. The critical equations were developed by direct application of the Euler-Lagrange equations to the horizontal channel flow. If Craya's hypotheses are treated similarly, one readily obtains the same relations as Craya. The corresponding critical conditions are

(a) for minimum thrust,

$$\dot{z} = \frac{\rho_o}{g} \frac{d^2(q^2)}{dp^2}$$

(b) for minimum energy flux,

$$\dot{z} = \frac{\rho_o}{g} \frac{1}{q} \frac{d^2(q^3)}{d\rho^2}$$

These equations resemble equation (23):

$$\dot{z} = \frac{\rho_o}{g} \; q \frac{d^2 q}{d\rho^2} \qquad (23)$$

Equation (23) appears to lead to physically meaningful results, with plausible fluid behavior in the vicinity of the boundary of separation. The limiting value of Richardson's number, $R = \frac{1}{2}$, at the separation boundary suggests stability of the flow. On the other hand, the same development of horizontal flow in a channel by means of the equations derived from Craya's hypotheses appears to lead to stability problems. The assumption of minimum thrust leads to a Richardson's number of zero at the boundary of separation. The other assumption, of minimum energy flux, requires the Richardson's number to become negative. The former condition is not dynamically stable, and the latter is not even statically stable; i.e., a negative Richardson's number implies a density inversion ($\dot{z} > 0$). For these reasons, the extremal hypothesis of this paper is preferred to the assumptions of Craya.

PARTIAL DIFFERENTIAL EQUATIONS FOR THE STREAM FUNCTION

General

Combining equations (9) and (15) yields

$$\rho_o G = g(Z-z) \frac{d\rho}{d\psi} \tag{42}$$

In general, solution of the critical equations will permit both Z and $\frac{d\rho}{d\psi}$ to be expressed as a function of ψ.

The channel and axisymmetric flow equations from (10) and (11) assume the following form:

for channel flow,

$$\rho_o \nabla^2 \psi = g[Z(\psi) - z] \frac{d\rho(\psi)}{d\psi} \tag{43}$$

for axisymmetric flow,

$$\rho_o \left(\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} \right) = \frac{gr^2}{\rho_o b^2} [Z(\psi) - z] \frac{d\rho(\psi)}{d\psi} \tag{44}$$

Stream Function Equations for Linear Source Stratification

With linear stratification of the source, from (37) and (41),

$$\frac{d\rho}{d\psi} = \frac{1}{\dot{\psi}} = - \frac{\pi}{h} \left(\frac{\rho_o}{gk} \right)^{\frac{1}{2}}$$

and

$$Z = \frac{\psi}{\psi_o} h$$

the channel equation becomes, from (43),

$$\nabla^2 \psi = - \frac{g}{\rho_o} \left[\frac{\psi}{\psi_o} h - z \right] \frac{\pi}{h} \left(\frac{\rho_o}{gk} \right)^{\frac{1}{2}}$$

Hence, from (39),

$$\nabla^2 \psi = - \frac{\pi^2}{h^2} \left(\frac{\psi}{\psi_o} - \frac{z}{h} \right) \psi_o$$

It is convenient to define dimensionless quantities as follows:

$$\Psi = \frac{\psi}{\psi_o} \qquad \eta = \frac{z}{h} \qquad \xi = \frac{x}{h}$$

In dimensionless form, the above relation for $\nabla^2\psi$ becomes

$$\frac{\partial^2\Psi}{\partial\eta^2} + \frac{\partial^2\Psi}{\partial\xi^2} = -\pi^2(\Psi-\eta) \tag{45}$$

Equation (45) is very similar to the dimensionless equation considered by Yih.[5] The two are identical if Yih's F is set equal to $\frac{1}{\pi}$.

The same procedure yields for the axisymmetric equation via (44):

$$\frac{\partial^2\Psi}{\partial\eta^2} + \frac{\partial^2\Psi}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial\Psi}{\partial\xi} = -\frac{\pi^2}{\beta^2}\xi^2(\Psi-\eta) \tag{46}$$

where the quantities are defined as:

$$\Psi = \frac{\psi}{\psi_o} \qquad \eta = \frac{z}{h} \qquad \xi = \frac{r}{h} \qquad \beta = \frac{b}{h}$$

The interpretation of b and β is as follows: In (44), if the flow were constrained to be horizontal at radius r = b, the velocity would be given by

$$q_z = 0$$

$$q_r = -\frac{b}{r}\frac{\partial\psi}{\partial z} = -\frac{\partial\psi}{\partial z}$$

In other words, if ψ is interpreted as the flow per unit arc length at the radius b, the horizontal flow at that radius would have the same character as the channel flow, where the horizontal velocity is $q_x = -\frac{\partial\psi}{\partial z}$. Thus, the total flow for the axisymmetric case (44) is the product of the periphery of radius b and the total critical flow per unit channel width, ψ_o

$$Q = 2\pi b\psi_o \tag{47}$$

where Q is the total flow to the axisymmetric sink. This does not mean that the axisymmetric flow is necessarily horizontal at radius b — in general, it is not; nevertheless, the total flow is given by (47) if ψ and ψ_o are taken from the critical equations based on horizontal channel flow. This is satisfactory if the assumption that the flow-density distribution, $\frac{d\rho}{d\psi}$, is indeed independent of the flow geometry. Thus "b" may be dubbed the "critical radius." The quantity β is the dimensionless counterpart of b and may be considered the critical radius, measured as a fraction of the flow height h.

CHANNEL FLOW TO A LINE SINK

Statement of Problem with Boundary Conditions

Consider the solution of the channel equation:

$$\frac{\partial^2\Psi}{\partial\eta^2} + \frac{\partial^2\Psi}{\partial\xi^2} = -\pi^2(\Psi-\eta) \tag{45}$$

The geometry may be considered as a long rectangular channel with the linearly stratified source fluid far upstream (Figure 2). A line sink is located at the base of a vertical dam at $\xi = 0$, $\eta = 0$. The bottom of the channel is the plane $\eta = 0$. Fluid in the channel extends to a height in excess of $\eta = 1$. In dimensionless coordinates, the stagnant source fluid is represented as

$$\Psi = \eta \tag{47}$$

This equation must be considered to apply in a region external to the region of steady flow (in steady flow, $\Psi = \eta$ would imply a uniform, horizontal velocity $-\frac{\partial\Psi}{\partial\eta} = -1$, which would contradict the assumption that the fluid is at rest). The total channel flow per unit width is determined by the critical equations so that ψ_o is already known prior to the solution of (45). (ψ_o corresponds to $\Psi = 1$). The additional information obtainable from the solution is the flow pattern of the streamlines, and detailed information on velocity and pressure, if desired.

The boundary conditions for (45) must now be considered. It is in this respect that the solution obtained here differs from Yih et al.[5] The following boundary conditions are straightforward:

$$\Psi(0,\eta) = 1 \qquad \Psi(\xi,0) = 0$$

The condition that applies at the upstream entrance of steady flow is somewhat arbitrary. If it is assumed that the upstream boundary is remote from the source fluid, it is reasonable to expect that the critical horizontal flow pattern will have been established in the channel. The condition that is naturally suggested is to require $\frac{\partial\Psi}{\partial\xi} = 0$ at the upstream boundary, which might be taken at any value of $\xi = \xi_1$ remote from the sink.

The final condition that must be specified relates to the upper boundary of the moving stream. A principal feature of the solutions for steady stratified flow developed in this report is that a fluid boundary separating the flowing stream from an overlying (or underlying) stagnant fluid must satisfy a condition of zero velocity. This is a consequence of requiring that the velocity be continuous across the boundary, as is the pressure. Since the stagnant fluid has zero velocity by definition, velocity at the boundary with the moving stream must be zero also. This is not to argue that discontinuous velocities cannot exist in fluids — the phenomenon is well known. The hypothesis here is that steady stratified flows derived from a rest condition by gravity will not create a velocity discontinuity and that there is no reason to assume that one exists. The picture that emerges from these considerations is that the upper boundary is a free fluid boundary such that

$$\left(\frac{\partial\psi}{\partial n}\right)_B = 0$$

$$\Psi_B = 1$$

n is the direction normal to the boundary curve.

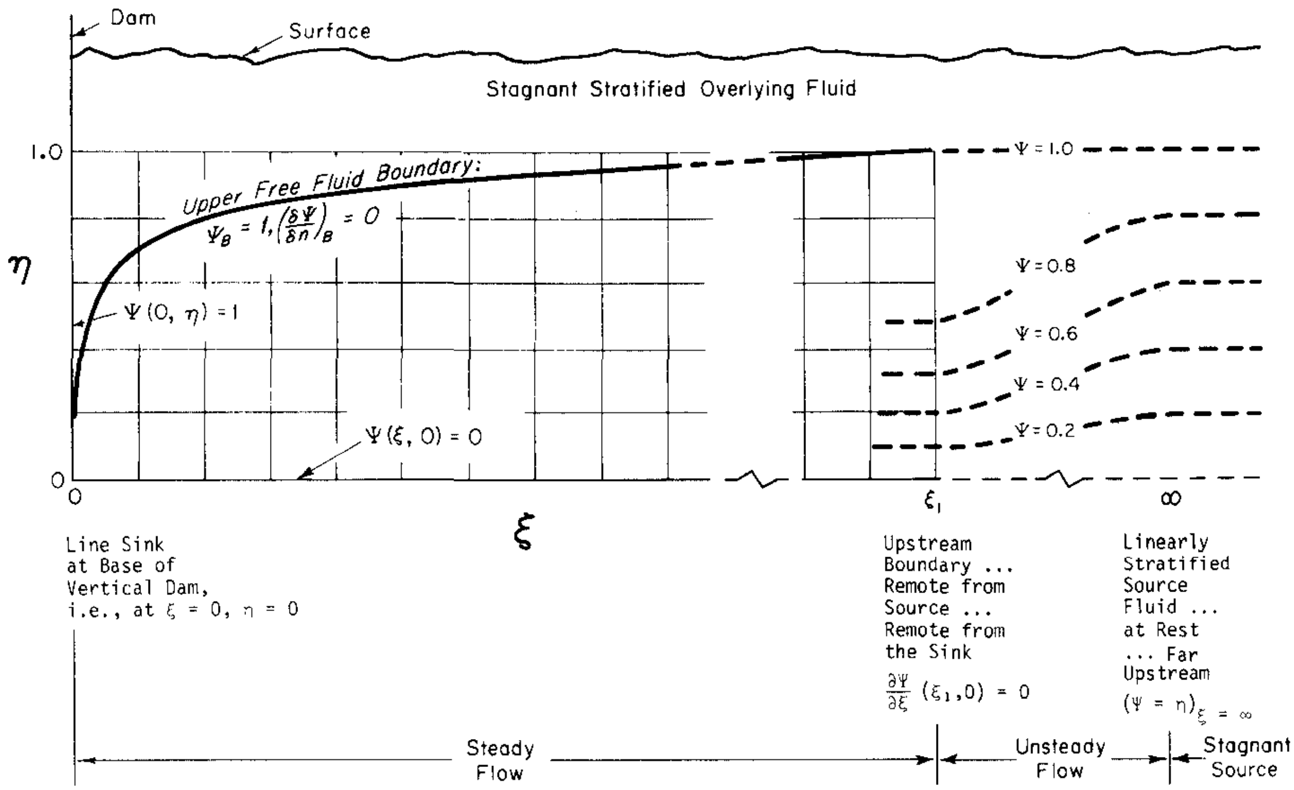


FIG. 2 CHANNEL FLOW TO A LINE SINK SHOWING CALCULATIONAL MESH AND BOUNDARY CONDITIONS

The free boundary condition just given is an example of a Cauchy, or third boundary problem in which both the scalar function and its normal derivative are given on the boundary. This is, in general, an overspecification if the boundary is itself a specified curve or surface in space. For a free boundary, the boundary curve is considered free to adjust to a (generally unique) curve such that the Cauchy condition may be satisfied. Suppose a steady stratified flow in a region "V" has an upper boundary curve of fluid separation represented by the equation $z = B(x)$ (Figure 3). If the overlying fluid is to be stagnant, and both velocity and density are to be continuous along the boundary, then the overlying fluid must be isopycnic in the region "S," which is shaded in Figure 3. The entire region S must have a uniform density equal to the density ρ_*, the layer bounding the moving stream; no other condition is hydrostatically stable. Above the maximum elevation z_m reached by the stream's boundary, stable stratification may exist, but will have no influence on the flowing stream.

Kao's model of stratified flow in a channel to a line sink[7] dealt with the upper boundary by assuming the formation of a stably stratified pocket of stagnant fluid above the boundary in a fashion qualitatively similar to that shown in Figure 3. However, his model retained the assumption that a velocity discontinuity exists at the boundary and implies the existence of discontinuities of density along the boundary. The stability of these conditions is questionable.

Wood[10] introduces the concept of a stable pocket or wedge that forms above a stratified moving stream. Hereafter in this paper, such a condition will be referred to as an "isopycnic wedge" or just "wedge."

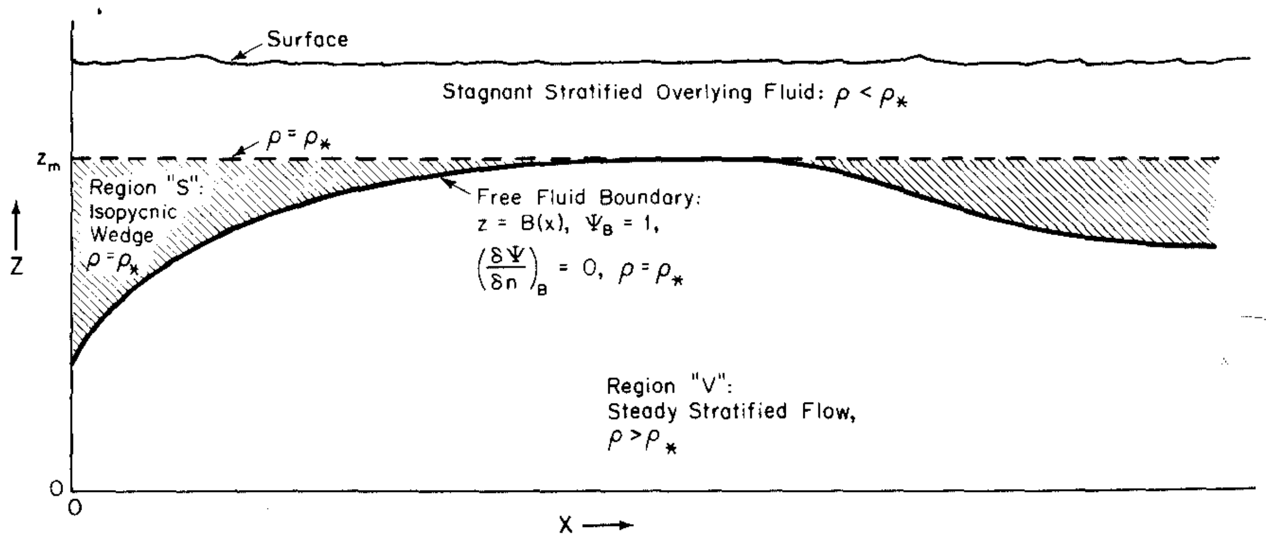


FIG. 3 FREE FLUID BOUNDARY SHOWING ISOPYCNIC WEDGES

In summary, the partial differential equation and boundary conditions are

$$\frac{\partial^2\Psi}{\partial\eta^2} + \frac{\partial^2\Psi}{\partial\xi^2} = -\pi^2(\Psi - \eta) \tag{45}$$

$$\Psi(\xi,0) = 0 \qquad \Psi(0,\eta) = 1$$

$$\frac{\partial\Psi}{\partial\xi}(\xi_1,\eta) = 0$$

$$\left(\frac{\partial\Psi}{\partial n}\right)_B = 0 \qquad \Psi_B = 1$$

B denotes the upper, free boundary of the stream. $\xi = \xi_1$ is the upstream boundary arbitrarily assumed for introduction of the steady flow.

It is easy to obtain the solution for Ψ in the horizontal flow remote from the sink. If indeed the streamlines are horizontal and have no curvature, then $\frac{\partial\Psi}{\partial\xi} = \frac{\partial^2\Psi}{\partial\xi^2} = 0$. Thus the vertical distribution of Ψ is readily obtained from (45):

$$\frac{d^2\Psi}{d\eta^2} = -\pi^2(\Psi - \eta)$$

The solution that meets the required boundary condition is:

$$\Psi = \eta + \frac{1}{\pi}\sin \pi\eta \tag{48}$$

It may be expected that equation (48) could be derived from the critical channel equations equally well. Indeed this may be done by combining equations (36) and (41) and recalling the definitions of the dimensionless variables. An alternative of the boundary condition $\frac{\partial\Psi}{\partial\xi} = 0$ at ξ_1 would be to require (48) to hold on the boundary $\xi = \xi_1$.

Solution of the Channel Equation

Equation (45) and the associated boundary conditions may be readily solved by iterative solution of the finite-difference equation. (Solution via separation of variables and harmonic expansion is complicated by the free boundary.) Interpretation of the boundary condition on the computational mesh is straightforward except for the Cauchy condition on the free (upper) boundary. A very satisfactory algorithm for handling the free boundary is based on the following features of the expected solution:

- In the flowing stream, Ψ may never assume a value greater than 1 (by definition of the boundary).
- The isopycnic wedge above the boundary, if it exists, must be represented by $\Psi = 1$ (i.e., same density as boundary layer).
- The free boundary curve must not extend above elevation $\eta = 1$. (The horizontal critical flow is confined to a height $\eta = 1$).

The algorithm is as follows: The condition $\Psi(\xi,1) = 1$ is imposed as a fixed boundary condition for the upper row of the lattice mesh. During the iterative calculation, whenever a value of Ψ is calculated for any interior lattice point, it is tested to determine if it exceeds 1. If it does not, the computed value of Ψ is entered and the computation proceeds to the next lattice point. If Ψ exceeds 1, the value of Ψ is entered as 1.0 for that lattice point, and the computation then proceeds to the next lattice point.

If a convergent solution is found by iteration with the above algorithm, a region will appear in which all lattice points have the value $\Psi = 1$. This region corresponds to the isopycnic wedge. Along the lower boundary of such a region, two successive values of $\Psi = 1$ in a given column (or row) correspond to a zero gradient of Ψ, which is just the condition that must hold if the velocity is to be zero at the boundary. If this account of the free boundary algorithm leaves the reader unconvinced, one can observe that, once a convergent solution is obtained, it is obviously possible to redefine the upper boundary of each column as the lowest elevation above the reference plane (smallest η) for which $\Psi = 1$ in this solution. It is apparent that the interior lattice points of this redefined problem will represent a valid, convergent solution of the redefined problem. It seems inescapable that, if there is a unique solution to the boundary value problem, the free boundary algorithm will provide a solution that will approach the true solution within the limitations of the finite mesh. By using a fine enough mesh, one could approximate the true boundary curve as closely as desired.

The channel problem was solved with a 5-point finite-difference formula for the operator ∇^2; in all cases, a mesh spacing of 0.05 was used in both ξ and η dimensions. The upstream boundary was taken as $\xi_1 = 10$. For the initial conditions, all interior lattice points were assigned the values specified by equation (48). With a Liebmann extrapolation constant in the range 1.8 to 1.9, convergent solutions were obtained in approximately 100 iterations. The criterion for convergence was that maximum change of Ψ for any mesh point must not exceed 10^{-5} for two successive iterations. The solution is shown in Figure 4. The streamline elevations were obtained by linear interpolation from the solution mesh. All solutions were computed on a UNIVAC 1108.

The following features of the solutions in Figure 4 are notable:

- In the vicinity of the upstream boundary (or entrance), $\xi = \xi_1$ the flow is indeed horizontal, and the functional relation between Ψ and η is close to equation (48) as expected. The slight difference may be attributed to the finite-difference approximation inherent in the 0.05 mesh. Little departure from this horizontal flow pattern is apparent until ξ is decreased to about 2.

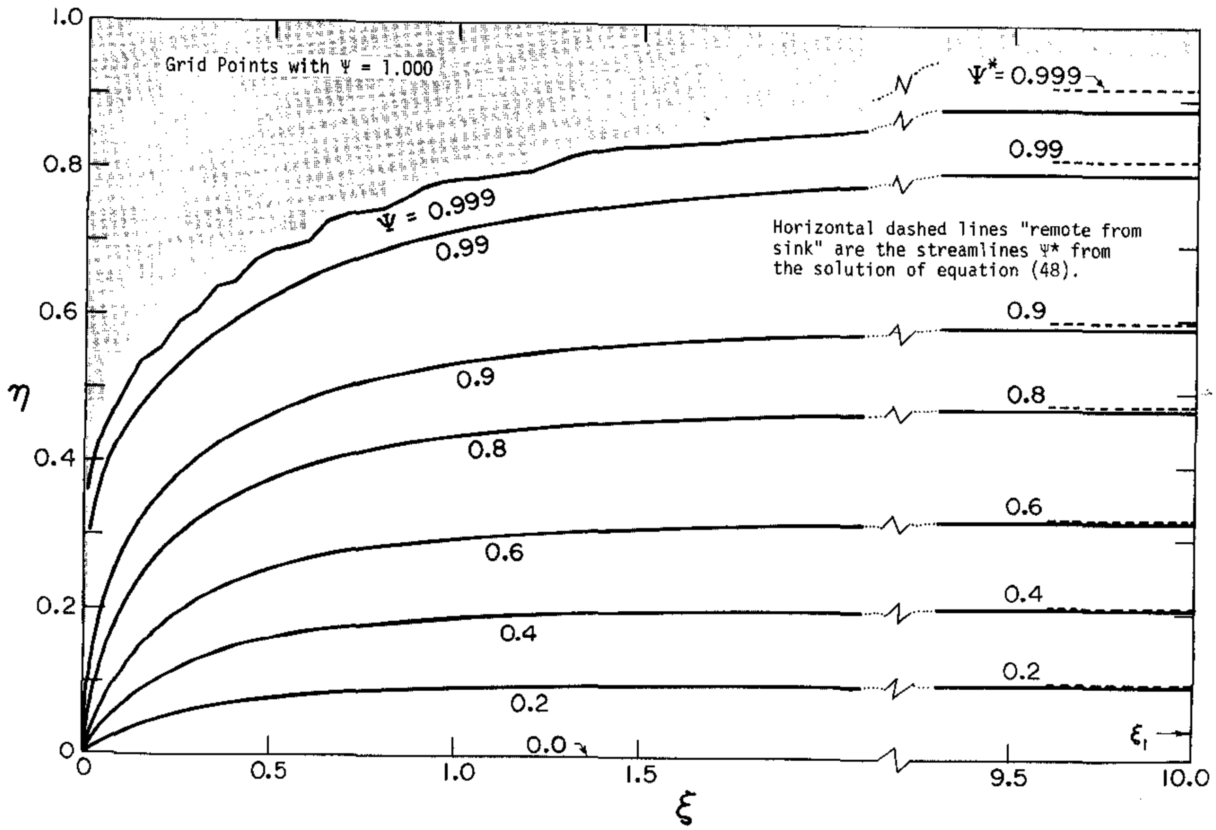


FIG. 4 STREAMLINE SOLUTION FOR CHANNEL FLOW

- The isopycnic wedge forms at the face of the "dam" ($\xi = 0$) and extends from $\eta = 1$ down to an elevation of about $\eta = 0.4$. At elevations less than $\eta = 0.4$ against the face, the velocity of the dividing streamline, $\Psi = 1$, is no longer zero (and the boundary is no longer free). Acceleration must occur as the fluid proceeds into the sink.
- The free boundary curve defining the lower extent of the isopycnic wedge approaches the line $\eta = 1$ asymptotically as ξ increases. The discontinuous nature of the curve is inherent in the finite mesh. These artificial boundary discontinuities influence adjacent interior streamlines somewhat, but seemingly damp out for streamlines $\Psi = 0.99$ or less.

The solution appears to be strikingly similar to the flow patterns obtained experimentally by Debler[6] (also shown in Reference 5).

AXISYMMETRIC FLOW

General Statement of Problem

Unlike two-dimensional flow in a channel, the equation for axisymmetric flow contains a parameter related to the total flow that must be determined by solution of the partial differential equation with boundary conditions and by treating the flow parameter as an eigenvalue in the solution. In channel flow, once the boundaries of separation are specified for a given source fluid, the channel flow per unit width is determined. Thus ψ_o is known prior to the solution of the partial differential equation. This point of view is extended to axisymmetric flow in the sense that ψ_o is predetermined in the same way; however, one does not generally know in advance what arc length to multiply ψ_o by to get the total flow. The critical radius b (or the dimensionless β) is an eigenvalue characteristic of the sink geometry.

The interpretation of β as a critical eigenvalue may be readily seen by the following: Consider the dimensionless axisymmetric equation,

$$\frac{\partial^2\Psi}{\partial\eta^2} + \frac{\partial^2\Psi}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial\Psi}{\partial\xi} = -\frac{\pi^2}{\beta^2}\xi^2 (\Psi - \eta) \tag{46}$$

where $\eta = \frac{z}{h}$ and $\xi = \frac{r}{h}$, $\Psi = \frac{\psi}{\psi_o}$, $\beta = \frac{b}{h}$

Visualize a flow satisfying (46) that is constrained to be horizontal at radius $\xi = \beta$. Horizontal flow is taken to mean that the streamlines are horizontal and have no curvature. Thus $\frac{\partial\Psi}{\partial\xi} = \frac{\partial^2\Psi}{\partial\xi^2} = 0$ at $\xi = \beta$. From equation (46) at $\xi = \beta$, $\frac{d^2\Psi}{d\eta^2} = -\pi^2(\Psi - \eta)$. This is exactly the same as the equation for horizontal flow in a channel and, with the boundary conditions, the solution is

$$\Psi = \eta + \frac{1}{\pi}\sin \pi\eta \tag{48}$$

again the same as the channel equation. In this contrived geometry, ψ_o is the flow per unit arc length on a circle of critical radius b.

The total flow is then equivalent to that of a channel with a flow height h and a width $2\pi b$, i.e., $Q = 2\pi b\psi_o$. The dimensionless total flow $Q' = \frac{Q}{\psi_o h} = 2\pi\beta$. While several sink geometries will be considered for axisymmetric flow, in all of them the sink will be located in the reference plane $z = 0$ (or $\eta = 0$), and the reference plane will form a fixed, frictionless bottom for the flow as in the case of channel flow. Fluid is considered to extend above the elevation h (or $\eta = 1$) and to be stagnant above h.

Equation (46) may be solved by iteration of the corresponding finite-difference equation. For all solutions discussed in this report, the mesh spacing in both dimensions is 0.05. The criterion for convergence is again a maximum change of 10^{-5} between successive iterations. The free boundary algorithm is the same as that described for channel flow. Typically, a Liebmann extrapolation constant of 1.8 was used and convergent solutions were obtained in 40 to 150 iterations, depending upon the boundary conditions. If the mesh region is defined for too large a radius, divergent behavior is encountered. This is apparently related to the finite mesh; the finite-difference equation becomes unstable if the radius is permitted to increase indefinitely and the mesh size remains fixed.

Several kinds of boundary conditions were explored for the upstream boundary where the flow enters the region defined by the problem mesh. If this upstream boundary is taken sufficiently far from the sink, the pattern of flow in the vicinity of the sink and the eigenvalue β are insensitive to the assumed entrance conditions. The upstream boundary condition that was selected as standard was to require $\frac{\partial \Psi}{\partial \xi} = 0$, as in the case of channel flow.

In the following, a solution will first be given for flow to a point sink in which the flow parameter β is fixed in advance ($\beta = 1$). The overall character of the solution is similar to other axisymmetric geometries and serves as a basis for discussing features that are common to all cases. Subsequently, solutions will be considered for a point sink with different values of β; this will set the basis for describing an algorithm for finding a specific β as a solution to an eigenvalue problem. Finally, different sink geometries will be considered in which β is determined for a range of sink parameters.

Flow to a Point Sink with $\beta = 1$

For flow to a point sink located at $\eta = 0$, $\xi = 0$, the boundary conditions for equation (46) are:

$$\Psi(0,\eta) = 1$$

$$\Psi(\xi,0) = 0$$

$$\frac{\partial\Psi}{\partial\xi}(\xi_1,\eta) = 0$$

$$\Psi_B = 1 \tag{49}$$

$$\left(\frac{\partial\Psi}{\partial n}\right)_B = 0$$

$$\Psi(\xi,1) = 1$$

As in the channel case, B denotes the free boundary. The last condition is specified to constrain the total flow within the upper boundary of the mesh region defining the problem. The relationship of this latter condition to the free boundary condition will be apparent in the discussion of the solution. Two solutions were obtained: one for the upstream boundary at $\xi_1 = 2$ and the other for $\xi_1 = 3$.

The solutions are shown in Figure 5. The streamlines are plotted by linear interpolation from the solution mesh. The following features are noted in Figure 5.

- The isopycnic wedge appears at a radius ξ of about 1, and extends outward with rapidly increasing depth.
- At the outer boundary, $\xi_1 = 2$, the entering horizontal stream is confined to a height of about $\eta = 0.3$; for an outer boundary $\xi_1 = 3$, the height is about $\eta = 0.2$.

- From a radius of about $\xi = 1$ inward, the free boundary $\Psi = 1$ is constrained by the boundary condition $\Psi(\xi,1) = 1$, and is not affected by the free boundary algorithm.
- Over an appreciable region in the vicinity of $\xi = 1$, the streamlines appear to be nearly straight lines. Each streamline undergoes an inflection. A theory related to these inflections is discussed later.
- The entrance streams for the two cases $\xi_1 = 2$ and $\xi_2 = 3$ quickly approach the same flow pattern as the sink is approached.

It is easy to visualize the effect of extending the position of the outer boundary ξ_1 to even greater radii. As a limiting case, one could think of a boundary at ∞. The free boundary curve, $\Psi_B = 1$, would approach the plane $\eta = 0$ asymptotically as one follows the stream outward. [It can be shown that the theoretical limiting velocity distribution at infinity has the form $q^2 = \pi^2 (1 - \Psi^2)$. Thus the maximum dimensionless velocity of the stream, with $\Psi = 0$, is π. This is the velocity one gets if the maximum hydrostatic ΔP is computed between a depth of h (or $\eta = 1$) in the source fluid and the same depth in the isopycnic wedge.] Regardless of the position of ξ_1 (if it is not close to 1) and for any reasonable assumed flow pattern at ξ_1, the character of the flow quickly approaches that of Figure 5. In general, the fluid tends to flow in a stream confined to a minimum distance from the reference plane (discharge elevation).

Consider again the behavior of the boundary $\Psi = 1$ for Figure 5. From a radius of about 1 inward, the boundary is not free, but rather is constrained by the boundary condition $\Psi(\xi,1) = 1$. With this boundary fixed, it is clear that the velocity at the boundary does not remain zero; if the velocity is no longer zero, then the pressure must decrease as the velocity increases and the elevation $\eta = 1$ remains constant (as required by Bernoulli condition). These conditions are

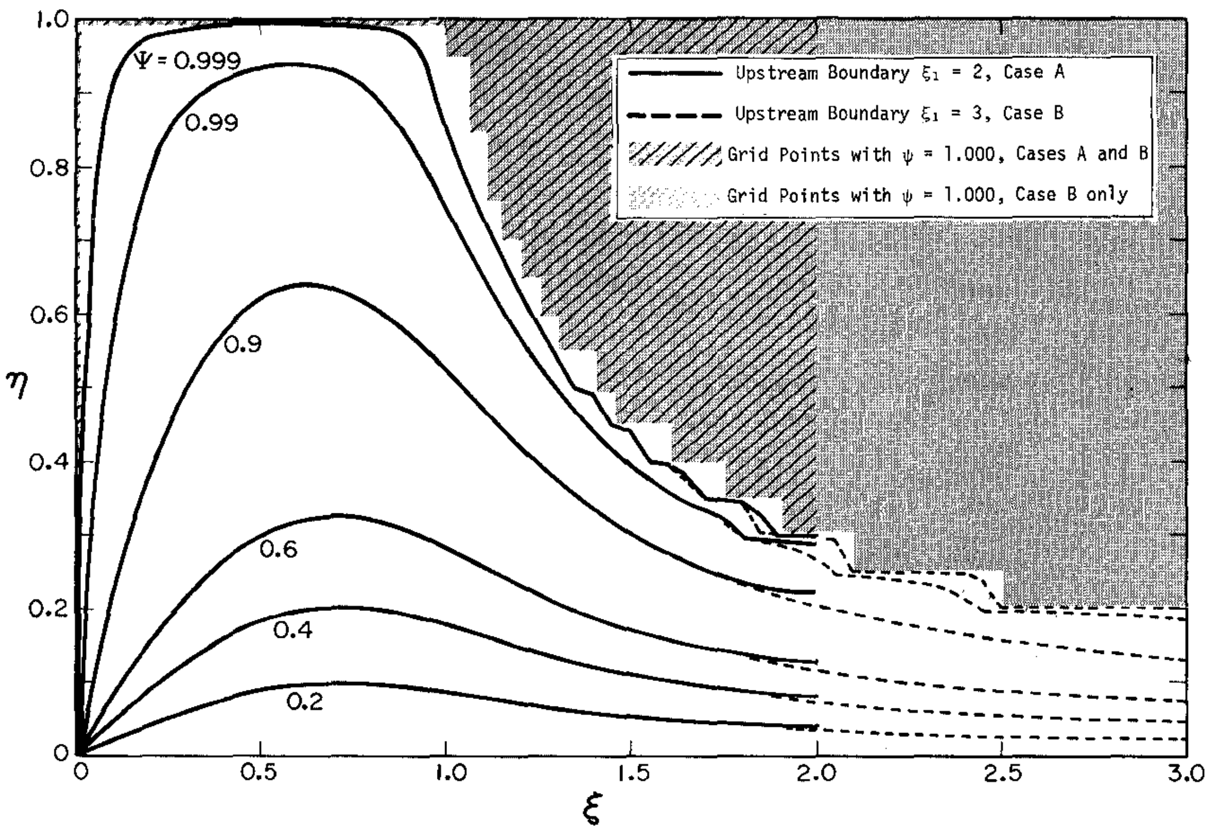


FIG. 5 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO A POINT SINK AT $\xi = 0$, $\eta = 0$ WITH CRITICAL RADIUS $\beta = 1$

incompatible with the existence of a fluid-to-fluid boundary in this region, if the overlying fluid is required to be stagnant. The resolution of the dilemma is that the boundary along $\eta = 1$ from $\xi = 0$ to $\xi = 1$ (approximately) must be considered a fixed, rigid boundary such as might be formed by a disk covering the region. With this concept, it is easy to explain how the eigenvalue β could be set equal to 1 in advance. The explanation is that the boundaries were not completely specified; i.e., the junction or attachment of the free boundary to the rigid disk was not specified. The radius of attachment, in effect, serves the function of an eigenvalue. Because of the finite mesh, the position of the free boundary attachment cannot be determined more closely than the 0.05 mesh spacing. The radius of attachment is taken as midway between the two columns that reflect the change from fixed to free boundary (Figure 6). For $\beta = 1$, the radius of attachment is $\xi_o = 0.975$.

Fixed Boundary at $\eta = 1.0$ ←; Radius of Attachment, $\xi = \xi_o = 0.975$ at $\eta = 1.0$

η \ ξ	0.80	0.85	0.90	0.95	1.00	1.05	1.10
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.95	0.9953817	0.9967602	0.9981553	0.9993482	1.0000000	1.0000000	1.0000000
0.90	0.9899679	0.9927522	0.9956223	0.9981873	0.9998923	1.0000000	1.0000000
0.85	0.9829454	0.9871717	0.9916347	0.9958129	0.9989629	1.0000000	1.0000000
0.80	0.9734745	0.9791651	0.9853418	0.9914128	0.9965760	0.9998057	1.0000000

Free Boundary

Axisymmetric Flow to a Point Sink at $\xi = 0$, $\eta = 0$.
Critical Radius $\beta = 1$.
See Figure 5.

FIG. 6 Ψ VALUES AT THE CALCULATIONAL MESH POINTS IN THE VICINITY OF $\xi = 1$, $\eta = 0$

A series of problems was solved with the same boundary conditions as for Figure 5, but with different values for β, and the following results were obtained:

TABLE 1

β	ξ_o
0.6	*a*
0.7	0.425
0.8	0.625
1.0	0.975
1.5	1.625
2.0	2.185

a. No attachment; i.e., the isopycnic wedge extends all the way to radius zero.

Apparently, for $\beta \geqslant 1$, the radius of attachment is close to $\xi_o = \beta$. For $\beta < 1$, ξ_o decreases more rapidly than β and goes to zero between $\beta = 0.6$ and $\beta = 0.7$. With these data, it is now easier to state the physical problem to which the date of Table 1 provide an answer: If a horizontal disk of radius ξ_o and height $\eta = 1$ covers a point sink at the origin, the dimensionless critical radius is β as given above.

Flow to an Isolated Point Sink

The most important problem of axisymmetric flow is that of an isolated point sink: i.e., the sink is located in the reference plane at elevation zero and no other rigid boundaries are nearby. One might consider the problem as a degenerate case of a cover disk at height $\eta = 1$, but with $\xi_o = 0$. The free upper boundary must extend all the way to radius zero. On the other hand, the desired solution should not yield a condition of over-stagnation above the sink. Evidently, a solution exists for some β such that the free boundary is tangent to the plane

$\eta = 1$ at a single point. Let this value of β be called β_o. For $\beta > \beta_o$, $\xi_o > 0$ a rigid cover is required to maintain the validity of the solution. For $\beta < \beta_o$, over-stagnation results. Physically, it is reasonable to expect the flow to an isolated sink to tend toward a pattern that yields $\beta = \beta_o$.

To determine the value of β_o, one could proceed by trial and error to extend the data of Table 1; however, this is cumbersome and computationally inefficient. It was possible to devise a simple algorithm to vary β during the iterative computation so that, in effect, the trial and error search for the eigenvalue β_o was carried out simultaneously with the development of the mesh solution for Ψ. In general, the critical eigenvalue is the one that causes the free boundary to be tangent to the plane $\eta = 1$ at a single point. It was not difficult to find algorithms that would find the critical eigenvalue with a precision of better than one percent. Eigen solutions for β were readily obtained, and the free boundaries of the solutions exhibited the desired characteristics.

For the isolated sink, the value of β_o was found to be 0.63. Figure 7 shows the streamlines for the solution. Figure 8 shows a portion of the solution mesh for Ψ in the region $\xi = 0$, $\eta = 1$. The close proximity of Ψ to 1 (but yet $\Psi < 1$) for the innermost, upper interior point strongly implies that the true curve of the free boundary is tangent to $\eta = 1$ at the origin $\xi = 0$, if the limitation of the finite mesh is removed.

The solution for axisymmetric flow to a point sink is especially important because it is a "self-similar" solution and reflects the limiting behavior of any sink of finite size. Even irregular structures forming an outlet for fluid being drawn from a large body of water will be well represented as point sinks when the flow is sufficiently great that the thickness of the withdrawn stream is large compared to the dimensions of the outlet structure

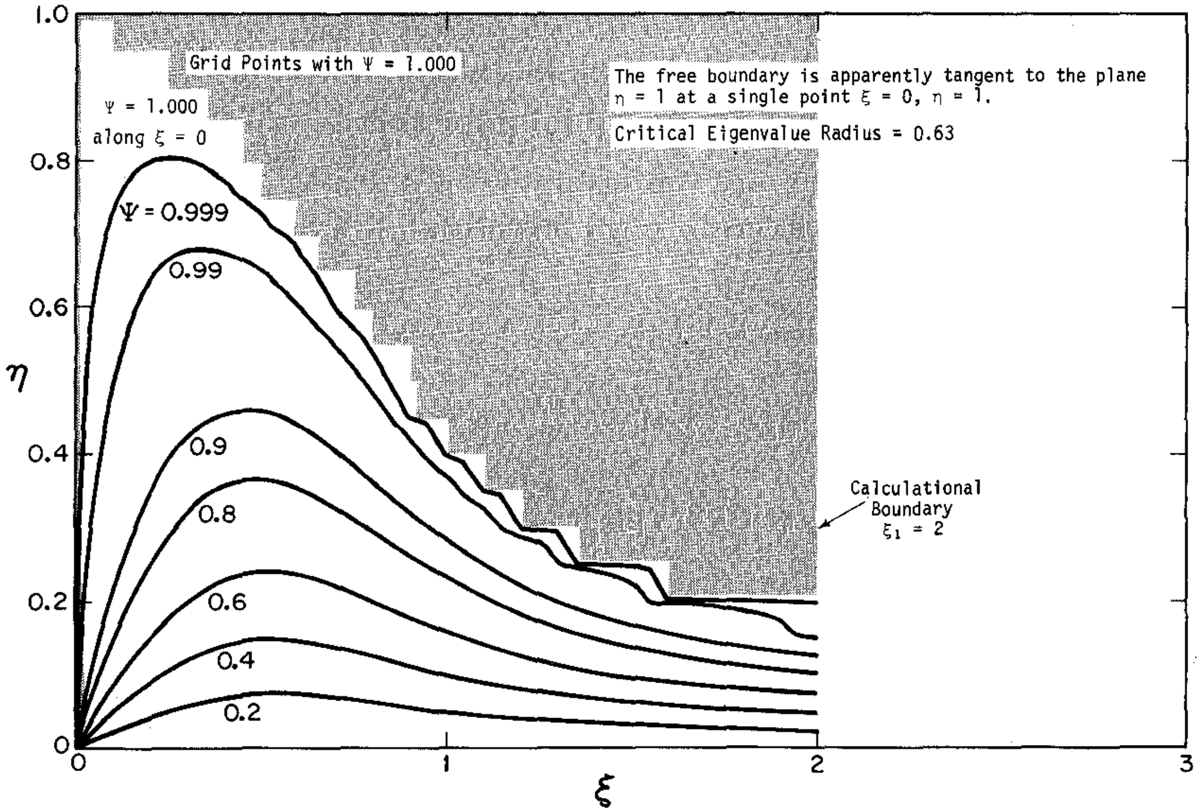


FIG. 7 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO AN ISOLATED POINT SINK AT $\xi = 0$, $\eta = 0$

η \ ξ	0	0.05	0.10	0.15	0.20
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.95	1.0000000	0.9999998	1.0000000	1.0000000	1.0000000
0.90	1.0000000	0.9999915	0.9999762	0.9999715	0.9999895
0.85	1.0000000	0.9999627	0.9998745	0.9997823	0.9997413
0.80	1.0000000	0.9998986	0.9996384	0.9993102	0.9990340

Axisymmetric Flow to an Isolated Point Sink at $\xi = 0$, $\eta = 0$.
Critical Eigenvalue Radius $\beta_0 = 0.63$.
See Figure 7.

FIG. 8 Ψ VALUES AT THE CALCULATIONAL MESH POINTS IN THE VICINITY OF $\xi = 0$, $\eta = 1$

(sink). For instance, a ring header of radius r_o in the reference plane will look like a point sink if the height of the flow, h, is much greater than r_o. It is possible to develop useful formulas for the point sink by adapting the critical equations obtained earlier. Recall equation (47) for the total axisymmetric flow:

$$Q = 2\pi b\psi_o \tag{47}$$

With $b = \beta_o h$ and equation (39) for a linearly stratified source,

$$Q = 2\beta_o h^3 \left(\frac{g}{\rho_o k}\right)^{\frac{1}{2}} \quad \text{or} \quad h = \left(\frac{\rho_o k Q^2}{4g\beta_o^{\ 2}}\right)^{\frac{1}{6}} \tag{50}$$

With $\beta_o = 0.63$,

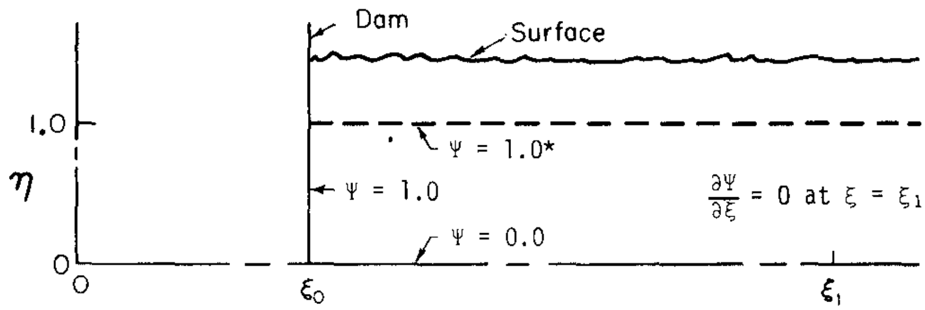
$$h = 1.082 \left(\frac{\rho_o k Q^2}{g}\right)^{\frac{1}{6}}$$

The depth of the stream, h, varies as the one-third power of the total flow to the sink.

Flow to Various Axisymmetric Sinks

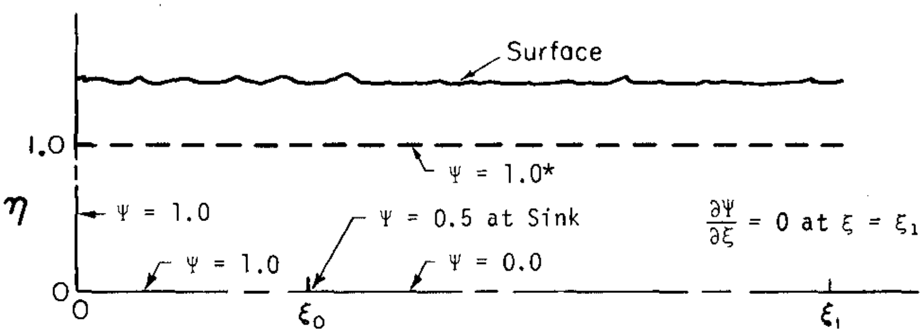
When the dimensions of a sink structure are comparable to the depth of the flowing stream, the behavior departs significantly from that of an isolated point sink. One example is that of a disk or cover above a point sink as discussed on page 47. Other cases of possible interest (Figure 9) are:

- Circular line sink in the reference plane at radius ξ_o
  - (a) Circular sink is located at the base of a dam in the form of a right circular cylinder
  - (b) Isolated circular sink
- Hole of radius ξ_o in the reference plane



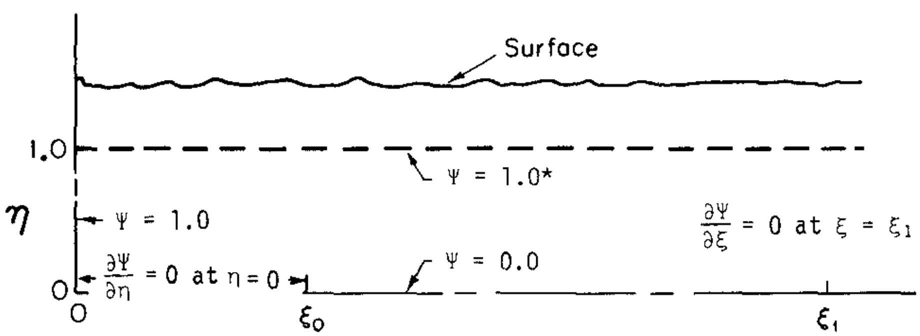
a. Circular Line Sink at Base of Dam ($\xi = \xi_0$)

See Figure 11



b. Isolated Circular Line Sink at $\xi = \xi_0$, $\eta = 0$

See Figure 12



c. Hole Sink from 0 to ξ_0 at $\eta = 0$

See Figure 13

* The upper boundary of the moving stream is also subject to the "free fluid boundary conditions." (See Figure 3)

FIG. 9 OTHER SINK GEOMETRIES WITH AXISYMMETRIC FLOW SHOWING BOUNDARY CONDITIONS

With appropriate boundary conditions, equation (46) was solved for different values of ξ_o to obtain the corresponding critical radius, β_o, in each case. The free boundary and upstream boundary conditions are the same as for the line sink. Care was taken to ensure that the upstream boundary, ξ_1, was far enough from the sink not to influence β_o significantly. The other boundary conditions appropriate for the sink geometries are:

Circular line sink with dam:

$\Psi(\xi_o,\eta) = 1$

$\Psi(\xi,0) = 0 \qquad \xi_o < \xi \leq \xi_1$

Isolated circular line sink:

$\Psi(0,\eta) = 1$

$\Psi(\xi,0) = 1 \qquad 0 \leq \xi < \xi_o$

$\Psi(\xi,0) = 0 \qquad \xi_o < \xi \leq \xi_1$

Hole:

$\Psi(0,\eta) = 1$

$\frac{\partial\Psi}{\partial\eta}(\xi,0) = 0 \qquad 0 \leq \xi < \xi_o$

$\Psi(\xi,0) = 0 \qquad \xi_o < \xi < \xi_1$

In the latter case, the hole radius ξ_o was considered to be midway between the mesh points. The requirement that $\frac{\partial\Psi}{\partial\eta} = 0$ at the hole is rather arbitrary.

Table 2 shows the values of β_o obtained for different ξ_o in the various geometries.

TABLE 2

A. Circular Line Sink at Base of Dam

Radius of Sink, ξ_o	Critical Radius, β_o
0.0	0.63
0.1	0.74
0.2	0.88
0.3	1.02
0.5	1.28
0.8	1.65
1.0	1.89
1.5	2.48
2.0	3.04
10.0	11.54

B. Isolated Circular Line Sink

Radius of Sink, ξ_o	Critical Radius, β_o
0.0	0.63
0.3	0.78
0.5	1.04
1.0	1.71
1.5	2.33
2.0	2.84

C. Hole Sink

Hole Radius, ξ_o	Critical Radius, β_o
0.0	0.63
0.325	0.76
0.525	0.97
1.025	1.62
1.525	2.29
2.025	2.85
3.025	4.00
4.025	5.09
10.025	11.40

The data are plotted in Figure 10. Typical streamline solutions are shown for $\xi_o = 1.5$ in Figures 11, 12, and 13. The following features are noted:

- In all cases the point of tangency of the free boundary to the plane $\eta = 1$ is at a larger radius than ξ_o.
- The isopycnic wedge for the circular sink at the base of a dam (Figure 11) extends to a depth of about $\eta = 0.75$. For greater ξ_o's, the profile of the wedge near the dam should approach that previously obtained for flow in a channel, since the channel case represents the limit of an infinite radius of curvature.
- The isopycnic wedges for the isolated circular sink and for the hole extend all the way to the reference plane in Figures 12 and 13. When ξ_o is small enough, however, the wedge does not reach the reference plane.

Neutral Surface

The streamline solutions for axisymmetric flow obtained in this paper have a characteristic pattern. Fluid entering at the upstream boundary, after a short distance of adjustment, appears to settle into a stream that decelerates until it begins to enter the field of the sink, where it accelerates. Each streamline appears to suffer an inflection where the streamline's curvature reverses; the inflection is in the region where the flow changes from deceleration to acceleration. Because of the inflection, the streamlines are approximately linear for an appreciable distance.

To test the credibility of the free boundary hypothesis and the consistency of the solutions obtained therefrom, an independent hypothesis and flow model were devised to explain the fluid behavior at the streamline inflections.

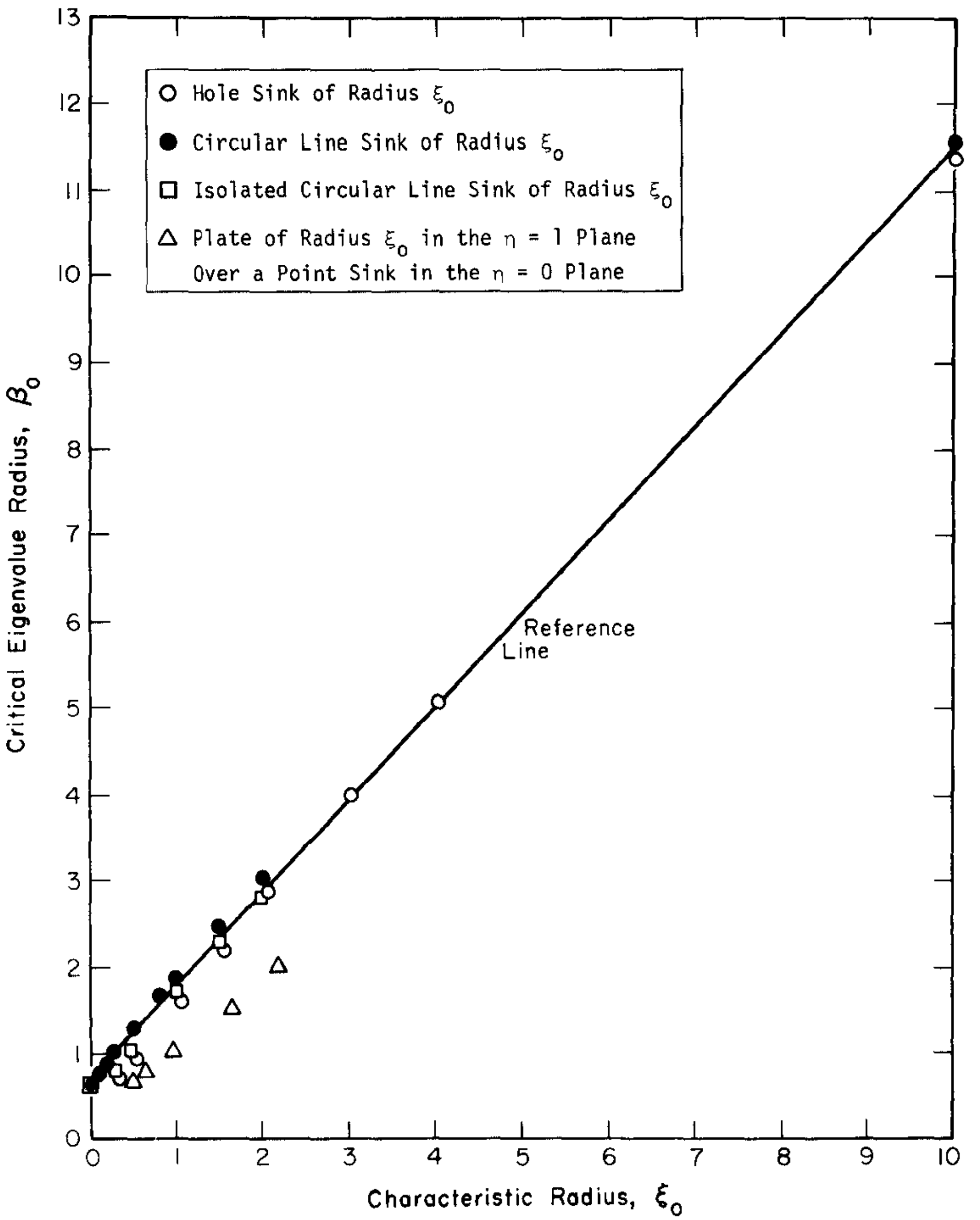


FIG. 10 CRITICAL EIGENVALUE RADIUS β_0 VERSUS CHARACTERISTIC RADIUS ξ_0

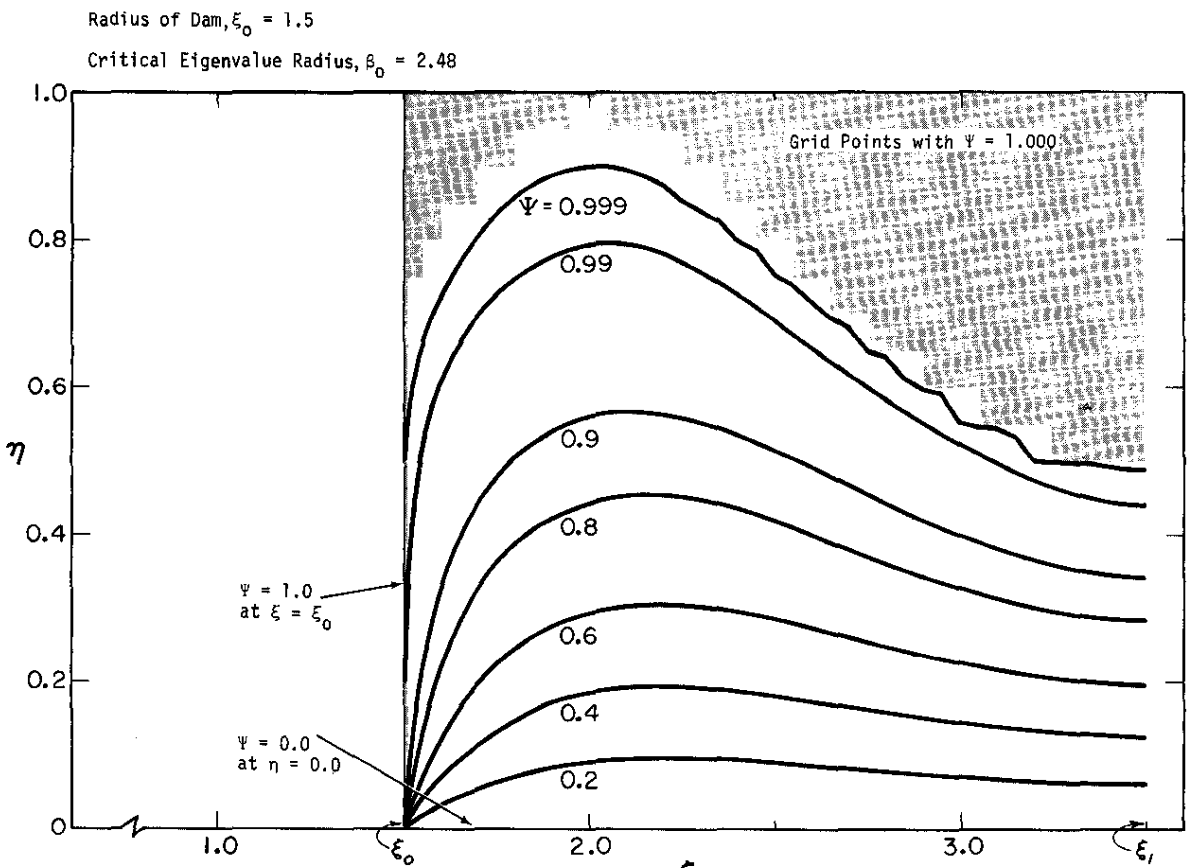


FIG. 11 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO A CIRCULAR LINE SINK AT THE BASE OF A DAM

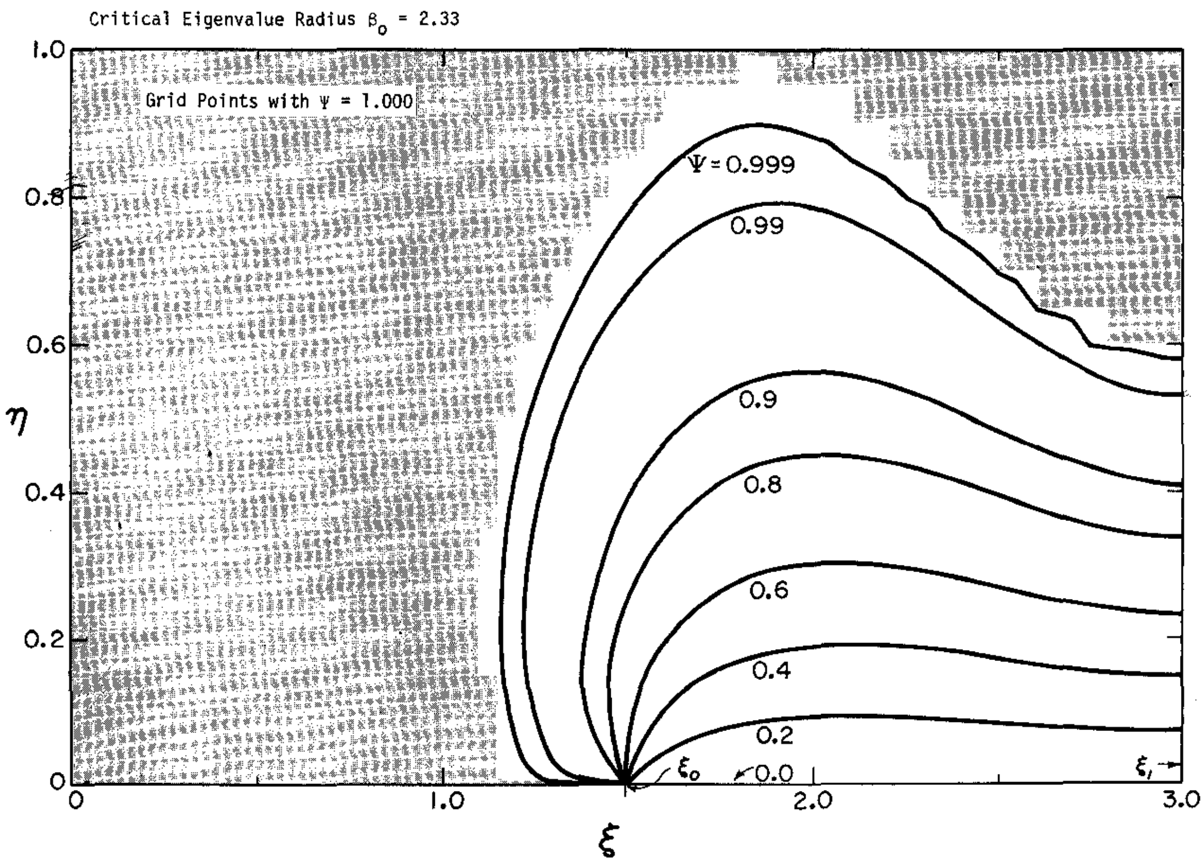


FIG. 12 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO AN ISOLATED CIRCULAR LINE SINK OF RADIUS ξ_0 = 1.5 IN THE η = 0 PLANE

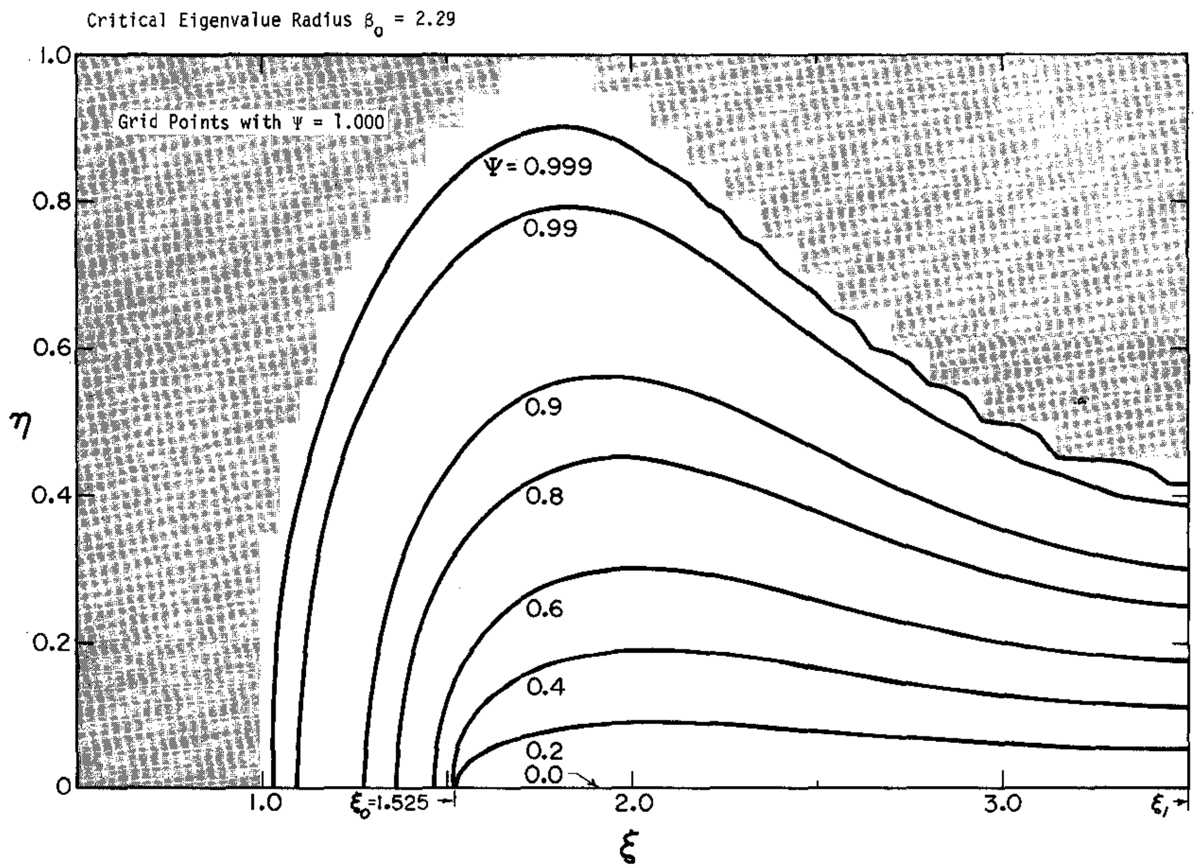


FIG. 13 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO A HOLE SINK OF RADIUS ξ_0 = 1.525 IN THE η = 0 PLANE

In the case of horizontal channel flow remote from source and sink, simple solutions were found for stratified flow on the premise that an extremal condition exists for gravity flow. The critical horizontal flow so derived has a characteristic that the fluid moves in a stable flow with zero acceleration. The pressure-depth relationship in any vertical section of the horizontal flow is the same as that calculated from the density-depth relationship on the basis of hydrostatic pressure. For axisymmetric flow, it is clear that such a condition of limiting flow in zero acceleration cannot persist indefinitely as it can in a channel of infinite length. However, it is possible to hypothesize the existence of such a limiting condition at a single, unique cross section in the flow. This is the concept of the neutral surface: Any stream of stratified fluid flowing under gravity from a source at rest through an appreciable region of steady-state flow with a free boundary will have a neutral, flow-normal surface. Properties of the fluid at the neutral surface are:

- Streamlines are orthogonal to the neutral surface.
- Fluid acceleration is zero.
- Streamlines undergo an inflection as they intersect the neutral surface.
- The pressure as a function of depth, along the neutral surface, is just the hydrostatic pressure characteristic of the density-depth relationship on the neutral surface.

Let the neutral surface be designated S_o. On this surface, the conditions (assumed as above) permit application of equations (17), (18), (19), and (20) just as for channel flow, if it is understood that

- q is the magnitude of the velocity of streamline with density ρ at S_o.
- z is the elevation of streamline ρ at S_o.

$Z(\rho)$ has the meaning it had before; namely, the elevation of streamline with density ρ at the source, where the fluid is at rest. With these interpretations, equation (20) holds on S_o:

$$\rho_o q \frac{dq}{d\rho} = g(Z - z) \tag{20}$$

It is convenient to define a dimensionless velocity q_* that is compatible with the dimensionless variables already in use. An obvious definition is

$$q_* = \frac{\text{Actual velocity}}{\text{Mean critical velocity in channel flow}}$$

where the denominator is simply $\frac{\psi_o}{h}$; thus

$$q_* = \frac{qh}{\psi_o} \tag{51}$$

This definition is compatible with the previous definitions as can be seen from the following: In the dimensionless axisymmetric equation (46), the magnitude of the dimensionless velocity should be

$$q_*^2 = \frac{\beta^2}{\xi^2} \left[\left(\frac{\partial\Psi}{\partial\eta}\right)^2 + \left(\frac{\partial\Psi}{\partial\xi}\right)^2\right] \tag{52}$$

where $\Psi = \frac{\psi}{\psi_o}$, $\eta = \frac{z}{h}$, $\xi = \frac{r}{h}$, and $\beta = \frac{b}{h}$; and where the actual velocity magnitude is given by

$$q^2 = \frac{b^2}{r^2} \left[\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right]$$

One gets by substitution:

$$q_*^2 = q^2 \frac{h^2}{\psi_o^{\,2}}$$

which agrees with (51). It is now possible to write (20) in dimensionless form with the help of (51), (37), (39), and (41). Thus upon the neutral surface, S_o:

$$\frac{d}{d\Psi}\left(\frac{q_*^2}{2}\right) = -\pi^2(\Psi - \eta) \qquad (53)$$

Now, in axisymmetric flow, S_o itself must be an axisymmetric surface, e.g., a surface of revolution about the axis $\xi = 0$. Let the generating curve for S_o be the curve T with arc length t measured from the reference plane. Since S_o was assumed to be a surface normal to the flow, the curve T is normal to the streamlines. Thus, on S_o:

$$q_*^2 = \frac{\beta^2}{\xi^2}\left(\frac{d\Psi}{dt}\right)^2 \qquad (54)$$

Equation (53) may be rewritten, via (54):

$$\frac{d}{dt}\left(\frac{1}{\xi}\frac{d\Psi}{dt}\right) = -\frac{\pi^2}{\beta^2}\xi(\Psi - \eta) \qquad (55)$$

If the generating curve of S_o were known, i.e., ξ and η were known functions of t, then (55) would relate Ψ and t in an ordinary second-order differential equation.

The nature of S_o follows logically from the assumption that the acceleration of the fluid is zero at S_o. If the acceleration is zero, the curvature of the flow-normal surface must also be zero. More precisely, if κ is the mean curvature, then $\kappa = 0$ everywhere on S_o. If r_1 and r_2 are the principal radii of curvature at a point on a surface, then:

$$\kappa_1 = \frac{1}{r_1}; \ \kappa_2 = \frac{1}{r_2}$$

r_1 and r_2 have opposite signs if their corresponding centers are on opposite sides of the surface. The mean curvature is

$$\kappa = \frac{1}{2}(\kappa_1 + \kappa_2)$$

Surfaces with zero mean curvature are well known in differential geometry. They are referred to as minimal surfaces because they are generally derived from a requirement that a surface have a minimum area subject to reasonable boundary constraints - e.g., soap-film models. The surface represented by a plane section normal to the horizontal channel flow is a minimal surface. For axisymmetric flow, the required minimal surfaces are surfaces of minimal area of revolution that are constrained to be normal to the reference plane. By standard variational methods, it can be shown that such minimal surfaces are of the form:

$$r = r_m \cosh \left(\frac{z}{r_m}\right)$$

where r and z are the ordinary cylindrical coordinates, and r_m is the intercept of a surface in the reference plane (z = 0). These surfaces indeed have zero mean curvature. In terms of the dimensionless coordinates for axisymmetric flow, the minimal surface is

$$\xi = \xi_m \cosh \left(\frac{\eta}{\xi_m}\right) \qquad (56)$$

With equations (56) and (55), it is now possible to derive the desired differential equation. The choice of independent variable is arbitrary, but to make it easier to compare the result with the channel case, the independent variable was selected as η. Note that

$$dt^2 = d\eta^2 + d\xi^2 \qquad (57)$$

With (55), (56), and (57), the following equation is obtained upon S_o:

$$\frac{d^2\Psi}{d\eta^2} - \frac{2}{\xi_m} \tanh \left(\frac{\eta}{\xi_m}\right) \frac{d\Psi}{d\eta} = - \frac{\pi^2}{\beta^2} \xi_m^2 \, (\Psi - \eta) \cosh^4 \left(\frac{\eta}{\xi_m}\right) \qquad (58)$$

Boundary conditions for (58) are (on S_o):

$$\Psi = 0 \quad \text{when} \quad \eta = 0 \quad ; \quad \frac{d\Psi}{d\eta} = 0 \quad \text{when} \quad \Psi = 1$$

The latter condition is the requirement that the velocity be zero at the boundary $\Psi = 1$. If the constants β and ξ_m are specified, (58) is readily solved by numerical methods. Because of the form of the boundary conditions, the finite difference equation was solved by an iterative method with an algorithm similar to the free boundary algorithm to take care of the conditions at $\Psi = 1$.

In comparing the solution for Ψ upon S_o with that obtained from the partial differential equation, it is apparent that the value of β in (58) must be the same in both. The value of ξ_m, theoretically, could be obtained by plotting the inflection points of the streamlines in the full solution. However, the inflections, since they involve second derivatives, are strongly influenced by the finite-difference mesh and therefore do not yield a clean, smooth curve. A value of ξ_m was chosen so that the curve for S_o appears to give the best overall fit to the inflection region. It is emphasized that only the single parameter ξ_m is selected to make the fit.

Figure 14 shows the curve for neutral surface S_o determined for flow to an isolated point sink; ξ_m has the value of 0.8. For values $\xi_m = 0.7$ and 0.9, the corresponding surfaces for S_o visibly departed from orthogonality with the streamlines. Figure 15 compares the agreement between Ψ determined from the partial differential equation and equation (58) in the case of the flow to a point sink. The agreement is good and affords support for the hypotheses of the neutral surface.

For an axisymmetric sink of very large radius, it may be expected that ξ_m and β will be approximately equal to the sink radius and the ratio ξ_m/β should approach 1 as larger and larger radii are considered. The limiting form of (58), as ξ_m approaches infinity and ξ_m/β approaches unity, is

$$\frac{d^2\Psi}{d\eta^2} = -\pi^2(\Psi - \eta)$$

This is the same as the governing equation for horizontal channel flow, as it should be, and yields the solution given by (48).

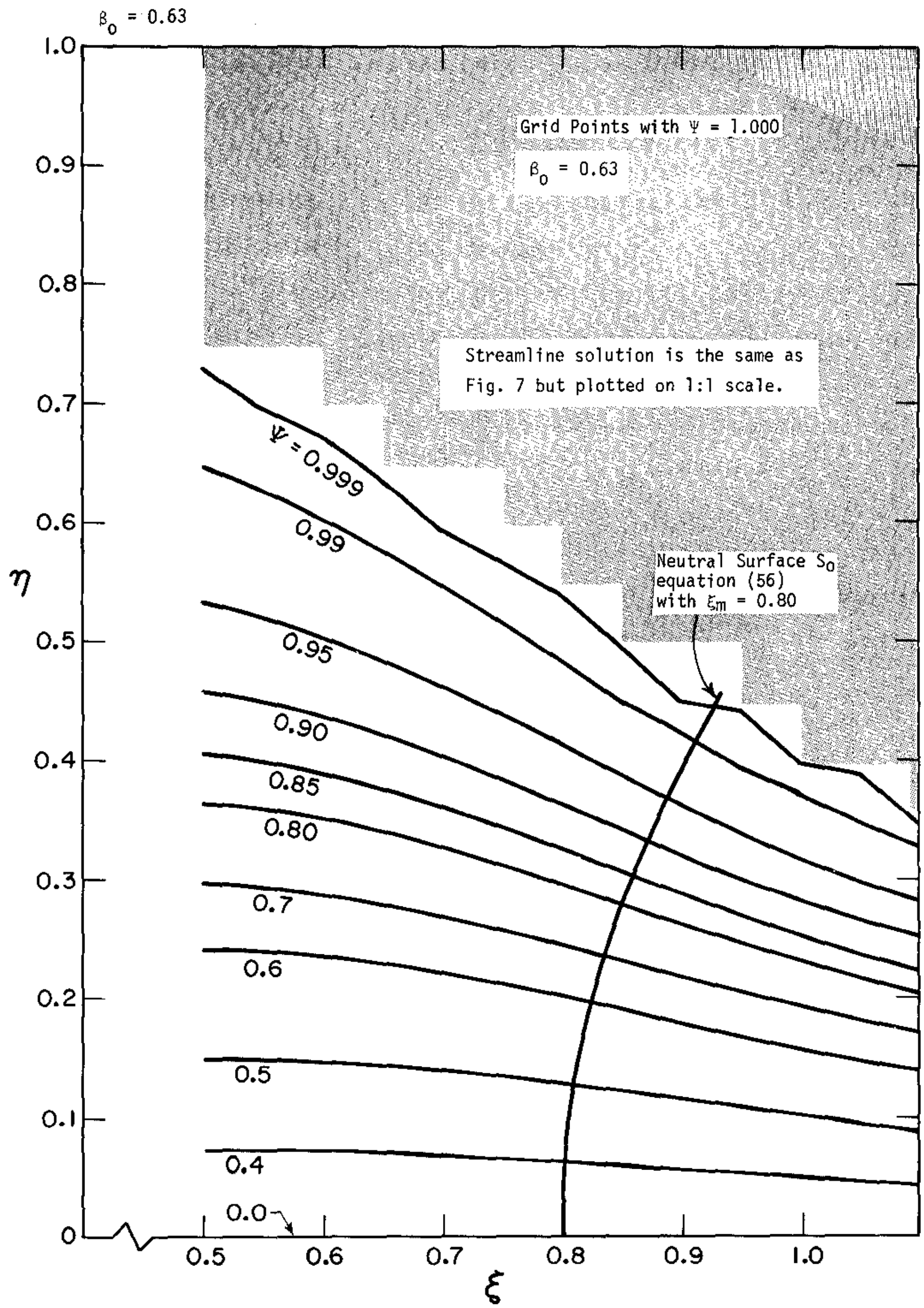


FIG. 14 STREAMLINE SOLUTION FOR AXISYMMETRIC FLOW TO AN ISOLATED POINT SINK AT $\xi = 0$, $\eta = 0$ SHOWING THE NEUTRAL SURFACE

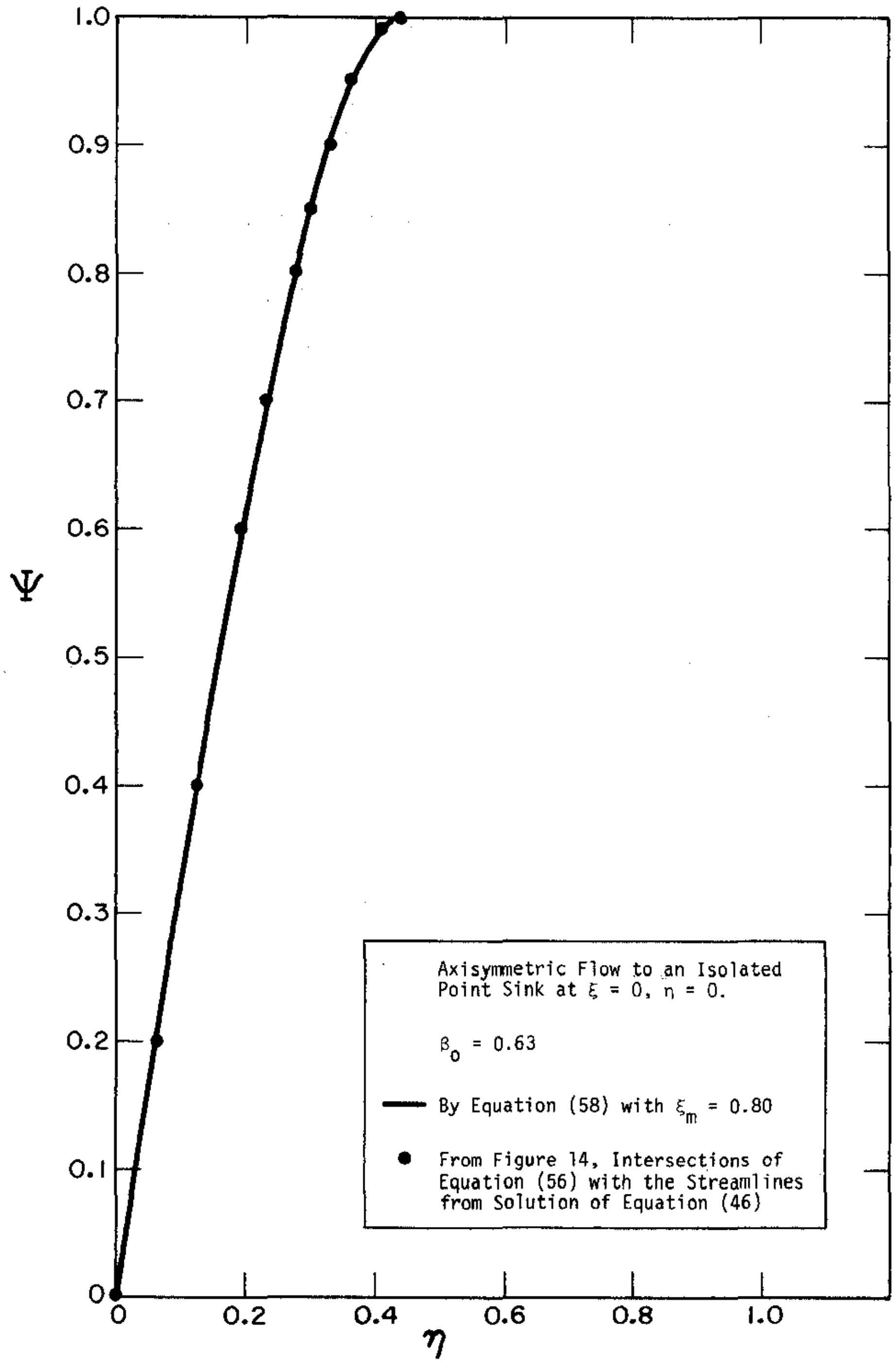


FIG. 15 COMPARISON OF Ψ VERSUS η ON THE NEUTRAL SURFACE

VELOCITY AND PRESSURE DISTRIBUTIONS

Once a solution for Ψ is obtained for a specific problem either in channel or axisymmetric flow from equations (45) or (46), it is easy to obtain the detailed velocity and pressure distributions, if desired. Since the streamlines themselves convey complete information on the direction of fluid flow, there is little need for the individual components of velocity, and the total magnitude is of primary interest. The dimensionless magnitude of the velocity for axisymmetric flow was defined in the discussion of the neutral surface as

$$q_*{}^2 = \frac{\beta^2}{\xi^2}\left[\left(\frac{\partial\Psi}{\partial\eta}\right)^2 + \left(\frac{\partial\Psi}{\partial\xi}\right)^2\right] \qquad (52)$$

For channel flow,

$$q_*{}^2 = \left(\frac{\partial\Psi}{\partial\eta}\right)^2 + \left(\frac{\partial\Psi}{\partial\xi}\right)^2 \qquad (59)$$

Thus q_* is readily computed from the solution mesh for Ψ by means of the finite-difference representations of (52) and (59). From (59) and (48), at the bottom of the horizontal channel flow ($\eta = \Psi = 0$), $q_* = 2$. The maximum velocity, at the bottom of the channel, is twice the mean velocity in the channel.

With knowledge of the velocity and fluid elevations along a streamline, it is straightforward to find the pressure. It is convenient to define a dimensionless quantity that contains the pressure information as a meaningful pressure differential. Let $P_1(z)$ be a reference hydrostatic pressure in a fluid that has (1) a density equal to the actual fluid at the boundary of separation, ρ_* (2) the same pressure as the actual fluid at ρ_* and elevation $z = h$ in the source. $P_1(z)$ is the pressure in the isopycnic wedge at elevation z; if the wedge does not extend to $z = 0$, $P_1(z)$ is still defined. A dimensionless pressure differential is defined as

$$P_* = \left(\frac{h^2}{\rho_o \psi_o^2}\right)(P - P_1) \tag{60}$$

where P is the pressure at any point in the fluid and P_1 is the "wedge pressure" (just defined) at the *same* elevation z. With equations (12), (17), (34), (39), (41), and (60) and $\eta = z/h$, the following expression for P_* is derived:

$$P_* = \frac{\pi^2}{2}\left[1 - 2\eta(1 - \Psi) - \Psi^2\right] - \frac{q_*^2}{2} \tag{61}$$

Equation (61) holds for either channel or axisymmetric flow from a linearly stratified source. In the limiting case remote from the sink where the isopycnic wedge approaches the reference plane and the fluid is confined to a stream of infinitesimal thickness, the maximum q_* (for $\Psi = 0$, $\eta = 0$, and $P_* = 0$) is $q_* = \pi$.

Computation of q_* and P_* was carried out for most of the streamline solutions, but the results are not reported here.

COMPARISON WITH EXPERIMENT

Detailed experimental data for flow of stratified fluids with continuously variable density do not appear to be readily available. The experiments of Debler,[6] Koh,[8] and Harleman et al.[9] afford some basis for comparison.

DEBLER's EXPERIMENTS[6]

Debler observed selective withdrawal of a linearly stratified fluid from a line sink at the bottom of a rectangular channel. He found that, for the flows and density gradients he used, the thickness of the withdrawal layer was associated with a modified Froude number F of about 0.24, were F was defined as

$$F = \frac{\psi_o}{h^2}\sqrt{\frac{\rho_o k}{g}}$$

Viscous effects would be expected to lower the experimental value of F. Debler suggested that the theoretical value of F for an ideal, inviscid fluid should be $1/\pi$, which is exactly the value predicted by equation (39). The flow patterns obtained by Debler with dyed fluid appear to be compatible with the solution given by Figure 4. Some qualitative differences in the patterns away from the vicinity of the sink could be explained by the difference in the finite tank used by Debler and the assumption of an infinite channel for the solution of Figure 4.

KOH'S EXPERIMENTS[8]

Koh's experiments were also conducted in the channel geometry with flow to a line sink. His objective was to obtain data with which to compare a theory that included friction contributions because of the viscosity of the fluid. As a consequence, his data were obtained under conditions where viscosity effects were not negligible. Thus Koh's stream thicknesses were greater than predicted by equation (39) and from Debler's correlation.

Koh was able to obtain good measurements of the velocity distribution in the flowing stream. Figure 16 shows Koh's normalized velocity distribution together with that derived from the cosine-squared distribution of equation (36). The agreement is good.

DATA OF HARLEMAN, MORGAN, AND PURPLE[9]

These data were obtained for axisymmetric flow of a stratified fluid to a sink of one-inch diameter. The source stratification was obtained by superimposing fresh water over a solution of sodium chloride with a procedure designed to minimize interfacial mixing. The position of the interface was defined and made visible by including a few droplets of a solution of di-n-butyl phthalate and xylene so proportioned that its density was midway between the fresh water and the underlying salt water. In each run, flow to the sink was increased until the interface dipped to indicate some flow at or above the interface into the sink.

The results of the experiments were correlated by relating the data to an approximate analytical expression that was obtained for flow to a point sink on the basis of assuming a discontinuous velocity between the flowing salt water stream and the stagnant overlying fresh water. This analysis yielded the following:

$$\frac{Q_c^{\,2}}{g' y_o^{\,5}} = 6.32\ K^2 = 2.58 \qquad (62)$$

where Q_c is the critical flow to the sink that draws down the interface, $g' = \frac{\Delta\rho}{\rho} g$; $\Delta\rho$ is the density difference between the two fluids of mean density ρ; y_o is the height of the interface above the reference plane; and K is a constant dependent upon the streamline geometry in the vicinity of the sink. The data were well correlated by selecting K = 0.64.

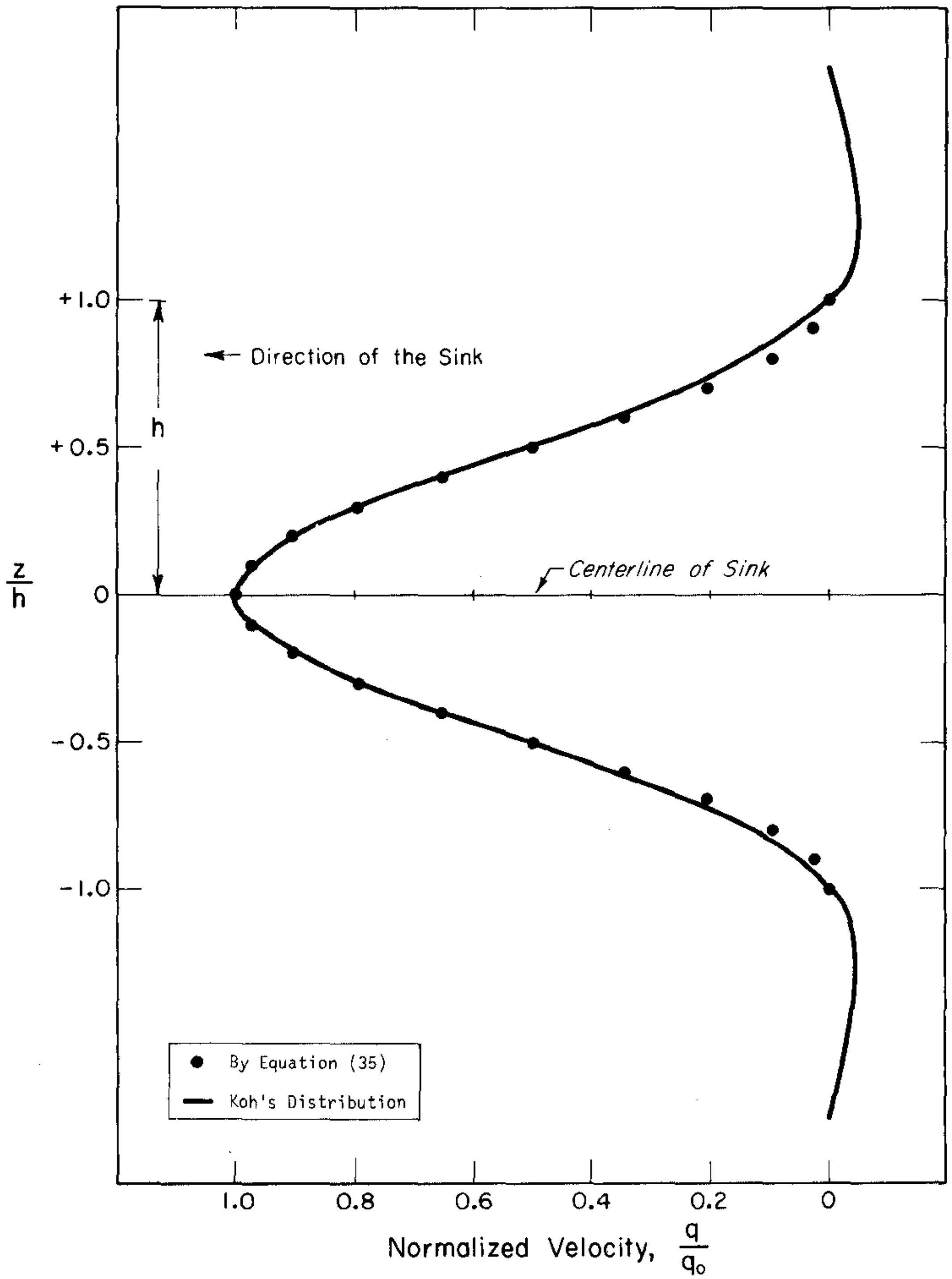


FIG. 16 KOH's NORMALIZED VELOCITY DISTRIBUTION VERSUS COSINE-SQUARED DISTRIBUTION EQUATION (35)

There are some uncertainties in comparing the analytical results obtained earlier in this paper with the information contained in the Harleman paper. These include:

- Differences in boundary conditions remote from the sink and uncertainty in the approach to steady-state conditions in the experiments.
- The amount of flow from above the interface that corresponds to the critical drawdown condition of the experiments.
- Uncertainty about the source stratification.

Nevertheless, an interesting comparison can be made with some reasonable assumptions. Let f be the ratio of critical flow from the experimental stratification of source height $y_o = h$ relative to that of a linearly stratified source that has a density span of $\Delta\rho/2$ over a height h. The value of k for the reference linear stratification is thus $k = 2h/\Delta\rho$. The "reference" flow of this linear source is thus, by (50),

$$Q = 2\beta_o h^3 \left(\frac{g\Delta\rho}{2\rho_o h}\right)^{\frac{1}{2}}$$

The actual (experimental) flow is $Q_c = fQ$; thus,

$$\frac{\rho_o Q_c^{\ 2}}{\Delta\rho g h^5} = \frac{Q_c^{\ 2}}{g' y_o^{\ 5}} = 2f^2\beta_o^{\ 2} \qquad (63)$$

If it is assumed that β_o is unaffected by the nature of the source stratification, the appropriate value for a point sink is $\beta_o = 0.63$ (Figure 7). There is a similarity in the concept of K as employed in the Harleman paper and β_o in this report. The comparison of K = 0.64 experimentally determined by Harleman with the $\beta_o = 0.63$ is striking.

A suitable value of f in equation (63) must now be found. Figure 3 of Harleman's paper shows a profile of the source stratification about the interface. If it is assumed that this profile is representative of all experimental runs, the critical flow and thus f relative to a linearly stratified source could be calculated for each y_o in the experiment via equations (24), (26), and (29). In view of the uncertainties, a simpler course was chosen. It was assumed that the source distribution is represented by a linear variation of density over a fraction γ of the source height (upper portion) and that the density is uniform over the remaining fraction ($1-\gamma$). With this assumption, f may be calculated as a function of γ; the result is shown in Figure 17. f has a maximum of 1.64 for $\gamma = 0.09$. Over the range $0.04 \leqslant \gamma \leqslant 0.18$, f is greater than 1.60. A γ of about 0.1 appears to represent the profile shown by Harleman for y_o (or h) of about 2 inches, which is about the median y_o for the experiments. It seems reasonable to take f = 1.6 in equation (63) to obtain (with $\beta_o = 0.63$):

$$\frac{Q_c^{\ 2}}{g' y_o^{\ 5}} = 5.09\ \beta_o^{\ 2} = 2.01 \qquad (64)$$

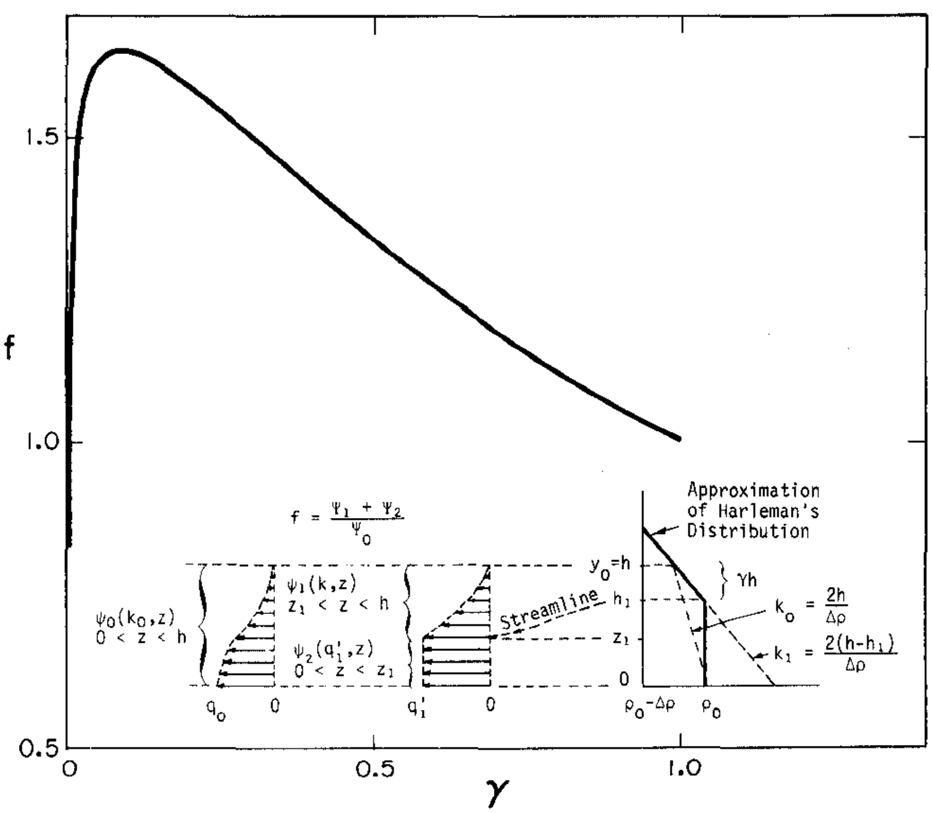


FIG. 17 CRITICAL FLOW RATIO f VERSUS FRACTION γ OF THE SOURCE HEIGHT

The agreement of (64) with (62) is surprisingly good. The difference in terms of the flow Q is about 12%, or in terms of the interface height, about 5%. For the lowest values of y_o in the Harleman data, it might be expected that f would decrease because the zone of interfacial density variation is greater relative to y_o. However, this should be offset by an increase in β_o when the radius of the sink (hole) is a significant fraction of y_o. The lowest data point plotted by Harleman is for $y_o \cong 0.75$, with a sharp-edged intake, for which the effective diameter is 0.75 inch (coefficient of contraction was 0.75). Thus the appropriate value of ξ_o for the hole-geometry of Figure 10 is about 0.5, with a corresponding value of $\beta_o \cong 0.95$. If no compensating decrease in f occurred, the constant on the right hand side of (64) would be about 4.5, which would yield a flow about 30% higher than (62).

Harleman in his Figure 7 plotted a "Dimensionless Interface Curve and Drawdown". In the nomenclature of this paper, he plotted η versus ξ for Ψ_I = constant, where Ψ_I represents a constant value of the stream function appropriate to the interface. No information is available on the amount of flow emanating from the region above Harleman's interface. Figure 18 shows Harleman's curve overlaid on the streamlines of Figure 7, which depicts flow from a linear source to a point sink. It appears Harleman's curve might correspond roughly to $\Psi = 0.98$, which would imply 2% of the flow comes from above the interface. Deviation of Harleman's interfacial streamline and that calculated in this paper may be expected as one moves away from the vicinity of the sink as a result of differences in boundary conditions and the extent to which steady flow is approached. Insufficient information is available to determine if the Harleman experiment offers any opportunity to confirm or deny the tendency to form an isopycnic wedge.

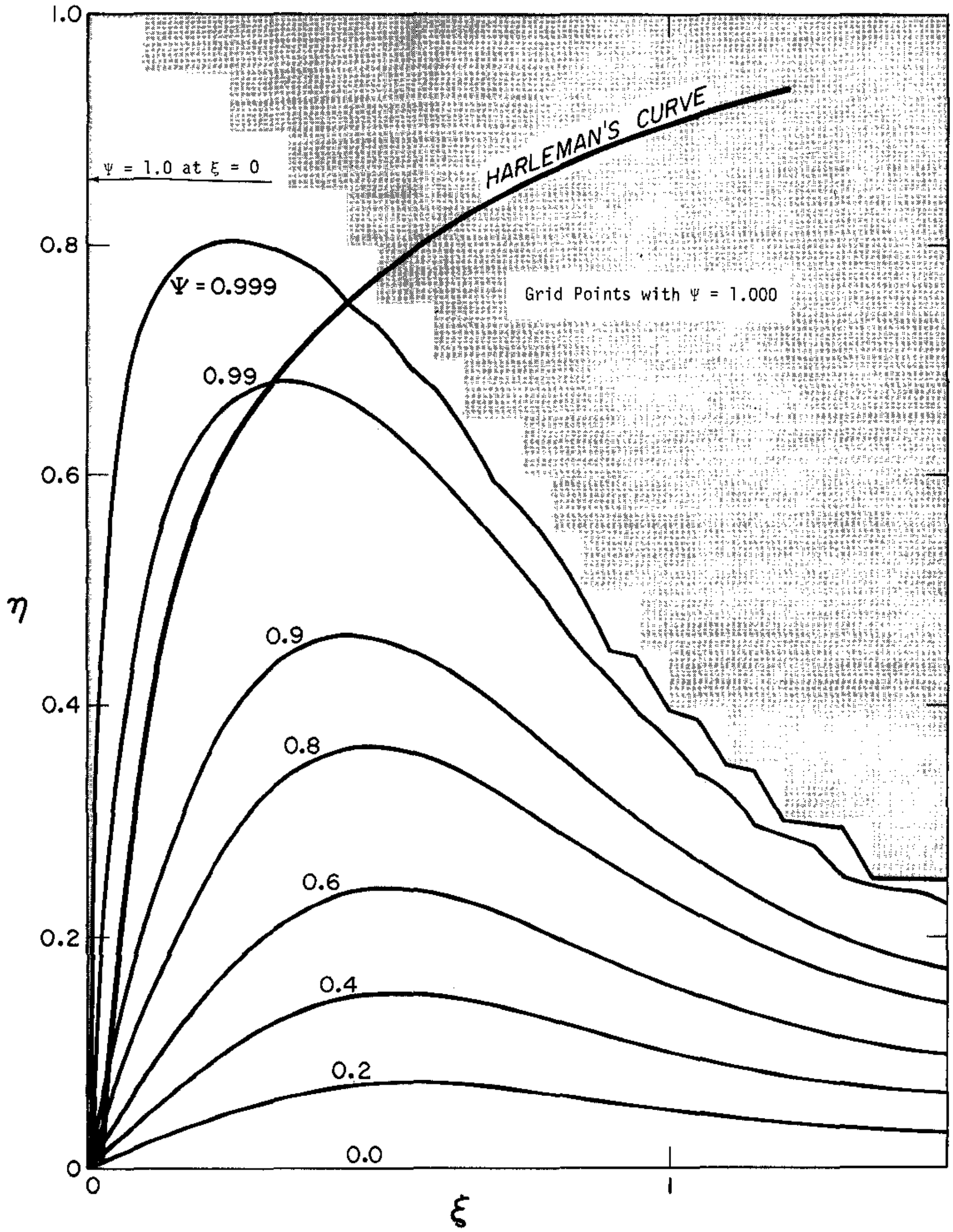


FIG. 18 HARLEMAN'S INTERFACE CURVE VERSUS CALCULATED STREAMLINES FOR AXISYMMETRIC FLOW TO AN ISOLATED POINT SINK*

* See Figure 7

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