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Physics

IBM 650 SUB-ROUTINES I

CIRCULAR AND HYPERBOLIC FUNCTIONS; REGULAR BESSEL FUNCTIONS

by

W. V. Baxter

Theoretical Physics Division

July 1955

E. I. du Pont de Nemours & Co.
Explosives Department - Atomic Energy Division
Technical Division - Savannah River Laboratory

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ABSTRACT

An IBM 650 sub-routine was programmed to compute the following functions: sin x, cos x, sinh x, cosh x; $J_n(x)$ and $I_n(x)$ for n = 0, 1, 2, and 3.

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INTRODUCTION

In reactor computations employing the IBM 650, it is desirable to have a sub-routine which will develop solutions to the diffusion equation $\nabla^2\phi + B^2\phi = 0$ in various geometries. In slab geometry, the solution to the equation is expressed in terms of circular and hyperbolic functions. In cylindrical geometry, the solution involves Bessel functions. The sub-routine for Bessel functions described in this report obtains only the regular functions, $J_n(x)$ and $I_n(x)$. The irregular Bessel functions $Y_n(x)$ and $K_n(x)$ are obtained by another sub-routine that will be published later.

SUMMARY

The sub-routine described here computes the functions sin x, cos x, sinh x, cosh x, $J_n(x)$, and $I_n(x)$ by series, using a variable number (N) of terms U_i, such that the last term U_N becomes $< 10^{-8}$. A fixed decimal argument is used to provide both fixed and floating decimal answers.

The programs occupy a block of 150 storages. Most of the programs are coded for minimum access time. The limitations on the values of the arguments of the functions are listed.

Two sets of coding are given: the first is a series listing for key-punching; the second is a detailed analysis for the individual functions.

DISCUSSION

Each of the functions developed by this sub-routine may be represented by a product $A(x)\ g(x)$, where $g(x)$ is the sum of an infinite series in $U_i(x)$, and $A(x)$ is defined as follows:

$$
\begin{aligned}
A(x) &= x && \text{for } \sin x,\ \sinh x \\
&= 1 && \text{for } \cos x,\ \cosh x \\
&= x^n \div 2^n n! && \text{for } J_n(x),\ I_n(x).
\end{aligned}
$$

Each $U_i(x)$ is obtained from the previous term, starting with $U_o = 1$. Using $r_i = r_{i-1} + \Delta r$, the formation of $g(x)$ is shown in the table:

Function	U_i = New Term	r_1 and Δr
$J_n(x)$	$-(\frac{x}{2})^2\ U_{i-1} \div r_i(r_i + n)$	1
$I_n(x)$	$(\frac{x}{2})^2\ U_{i-1} \div r_i(r_i + n)$	1
Sin x	$-x^2\ U_{i-1} \div r_i(r_i + 1)$	2
Sinh x	$x^2\ U_{i-1} \div r_i(r_i + 1)$	2
Cos x	$-x^2\ U_{i-1} \div r_i(r_i - 1)$	2
Cosh x	$x^2\ U_{i-1} \div r_i(r_i - 1)$	2

If the new term is less than 10^{-8}, the series is completed.

Use of Sub-Routine

The sub-routine is coded to occupy storages 0000 - 0149, and if translated m positions (m should be even) will occupy storages m - (0149 + m).

1. The argument of the function in the form xx.xxxxxxxx must be placed in storage (0004 + m).

2. A value of n must be placed in (0033 + m) in the form 000000000n.

3. An exit instruction must be placed in 0123, to return control to the main routine.

4. The fixed decimal answer will be found in the Lower Accumulator in the form xx.xxxxxxxx, and the floating decimal answer in the Upper Accumulator in the form yyx.xxxxxxx, where yy is the exponent (base 10) plus 50.

Function	n	Point of Entry	Range of Argument
sin x	1	0001	$\lvert x\rvert < 100$
cos x	-1	0001	$\lvert x\rvert < 100$
sinh x	1	0011	$\lvert x\rvert < 5.29$
cosh x	-1	0011	$\lvert x\rvert < 5.29$
$I_0(x)$	0	0031	$x < 6.32$
$I_1(x)$	1	0031	$x < 6.52$
$I_2(x)$	2	0031	$x < 6.77$
$I_3(x)$	3	0031	$x < 7.15$
$J_0(x)$	0	0101	$x < 7.82$
$J_1(x)$	1	0101	$x < 9.62$
$J_2(x)$	2	0101	$x < 8.94$
$J_3(x)$	3	0101	$x < 7.82$

The argument in 0004 and the value of n remain unchanged by the sub-routine. Thus if one function is computed, additional functions using the same argument or value of n do not require restoration of these quantities. The point of entry determines the new function if both x and n are the same as previously used. Of course, a new exit instruction must be inserted.

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PROBLEM: Sub-Routine in Storage Sequence

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0000	(00	0000	0000)	
0001	67	0004	0009	
0002	69	0005	0008	
0003	69	0108	0061	
0004	(00	0000	0000)	
0005	24	0069	0122	
0006	00	0000	0002	
0007	24	0013	0066	
0008	24	0106	0003	
0009	16	0012	0017	
0010	31	0001	0118	
0011	69	0014	0018	
0012	06	2831	8531	
0013	(00	0000	0000)	
0014	65	8003	0034	
0015	31	0001	0040	
0016	69	0004	0007	
0017	46	0020	0009	
0018	24	0069	0016	
0019	65	0013	0118	
0020	15	0012	0067	
0021	65	0013	(0068)	
0022	24	0021	0002	
0023	(00	0000	0000)	
0024	64	0006	0010	
0025	30	0007	0138	
0026	60	8001	0082	
0027	15	0131	0137	
0028	01	4000	0145	
0029				
0030	65	0033	0037	
0031	69	0014	0107	
0032	31	0001	0105	
0033	(00	0000	0000)	
0034	20	0039	0100	
0035	20	0139	0143	
0036	16	0093	0047	
0037	46	0049	0050	
0038	16	0041	0145	
0039	(00	0000	0000)	
0040	20	0045	0048	
0041	00	0000	0003	
0042	61	8002	0148	
0043	35	0001	0099	
0044	35	0001	0051	
0045	(00	0000	0000)	
0046	60	0000	0064	
0047	45	0135	0136	
0048	65	0033	0038	
0049	69	0052	0055	

PROBLEM: Sub-Routine in Storage Sequence (Continued)

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0050	69	0013	0055	
0051	64	0006	0015	
0052	01	0000	0000	
0053	11	0115	0027	
0054	20	0120	0129	
0055	24	0058	0021	
0056	65	0023	0105	
0057	24	0060	0063	
0058	(00	0000	0000)	
0059	20	0013	0066	
0060	(00	0000	0000)	
0061	24	0119	0030	
0062	31	0008	0083	
0063	24	0116	0119	
0064	10	0120	0065	
0065	21	0120	0074	
0066	69	0019	0022	
0067	10	0004	0076	
0068	35	0001	0024	
0069	(66)	8003	0034	
0070	15	0060	0125	
0071				
0072	65	0033	0087	
0073	69	0080	0084	
0074	60	0142	0096	
0075	65	0013	0068	
0076	46	0088	0059	
0077	15	0130	0123	
0078	24	0021	0073	
0079				
0080	24	0069	0072	
0081				
0082	19	8001	0102	
0083	45	0086	0091	
0084	24	0106	0111	
0085	46	0140	0042	
0086	20	0116	0070	
0087	45	0090	0095	
0088	11	0004	0141	
0089	24	0142	0046	
0090	16	0093	0097	
0091	60	0058	0114	
0092	31	0008	0113	
0093	00	0000	0001	
0094	60	8002	0104	
0095	65	0052	0105	
0096	10	0000	0134	
0097	45	0036	0056	
0098				
0099	64	0006	0032	

PROBLEM: Sub-Routine in Storage Sequence (Continued)

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0100	69	0103	0106	
0101	69	0004	0109	
0102	35	0002	0069	
0103	66	8003	0034	
0104	19	0116	0062	
0105	20	0058	0111	
0106	24	0069	(0072)	
0107	24	0069	0101	
0108	60	0006	0147	
0109	24	0013	0021	
0110	64	0139	0094	
0111	69	0052	0057	
0112	60	0023	0127	
0113	69	0117	0132	
0114	19	0060	0092	
0115	00	0000	0051	
0116	(00	0000	0000)	
0117	60	0093	0147	
0118	20	0023	0026	
0119	60	(0093)	0147	
0120	(00	0000	0000)	
0121	47	0126	0126	
0122	69	0075	0078	
0123	(00	0000	0000)	
0124	20	0130	0133	
0125	20	0060	0046	
0126	21	0131	0085	
0127	19	0045	0025	
0128	31	0001	0105	
0129	69	0033	0089	
0130	(00	0000	0000)	
0131	(00	0000	0000)	
0132	24	0119	0124	
0133	60	8002	0144	
0134	21	0142	0146	
0135	67	0039	0044	
0136	67	0039	0043	
0137	31	0002	0149	
0138	64	0041	0128	
0139	(00	0000	0000)	
0140	61	8002	0053	
0141	66	8002	0059	
0142	(00	0000	0000)	
0143	65	0039	0110	
0144	36	0000	0121	
0145	45	0028	0112	
0146	19	0120	0035	
0147	24	0000	0054	
0148	10	0115	0027	
0149	60	8002	0077	

PROBLEM: Initial Steps for $J_n(x)$

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0101	69	0004	0109	
0109	24	0013	0021	Put x into 0013.
0021	65	0013	(0068)	
0068	35	0001	0024	
0024	64	0006	0010	
0010	31	0001	0118	
0118	20	0023	0026	$x \div 2$
0026	60	8001	0082	
0082	19	8001	0102	
0102	35	0002	0069	
0069	(66)	8003	0034	
0034	20	0039	0100	$-(x \div 2)^2$
0100	69	0103	0106	
0106	24	0069	(0072)	Restore 0069.
				Enter A(x) routine.
0006	00	0000	0002	
0103	66	8003	0034	Constants
0004	(xx.	xxxx	xxxx)	Argument

PROBLEM: Modifications for $I_n(x)$

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0031	69	0014	0107	Set up 0069 to give $+(x \div 2)^2$.
0107	24	0069	0101	
				Enter $J_n(x)$ routine.
0014	65	8003	0034	Constant

PROBLEM: Form A(x) for Bessel Functions

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0072	65	0033	0087	
0087	45	0090	0095	Is n = 0?
0095	65	0052	0105	A(x) = 1
0105	20	0058	0111	Store A(x) in 0058: enter g(x)
				routine.
0090	16	0093	0097	
0097	45	0036	0056	Is n = 1?
0056	65	0023	0105	A(x) = x ÷ 2
0036	16	0093	0047	
0047	45	0135	0136	Is n = 2?
0136	67	0039	0043	
0043	35	0001	0099	
0099	64	0006	0032	
0032	31	0001	0105	$A(x) = (\frac{x}{2})^2 \div 2!$
0135	67	0039	0044	
0044	35	0001	0051	
0051	64	0006	0015	
0015	31	0001	0040	
0040	20	0045	0048	Store $(\frac{x}{2})^2 \div 2!$ in 0045.
0048	65	0033	0038	
0038	16	0041	0145	
0145	45	0028	0112	Is n = 3? Stop if not.
0028	01	4000	0145	
0112	60	0023	0127	
0127	19	0045	0025	
0025	30	0007	0138	
0138	64	0041	0128	$A(x) = (\frac{x}{2})^3 \div 3!$
0128	31	0001	0105	
0093	00	0000	0001	
0006	00	0000	0002	Constants
0041	00	0000	0003	
0033	00	0000	000n	
0023	(00	0000	0000)	x ÷ 2

PROBLEM: Formation of g(x) and Fixed Decimal Answer

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0111	69	0052	0057	
0057	24	0060	0063	Initial sum = 01.00000000.
0063	24	0116	0119	U_0 = 01.00000000.
0119	60	(0093)	0147	Δr = 1 or 2
0147	24	0000	0054	to 0000
0054	20	0120	0129	Clear 0120.
0129	10	0033	0089	
0089	24	0142	0046	$r_0 + n = n$ to 0142
0046	60	0000	0064	
0064	10	0120	0065	
0065	21	0120	0074	r_i
0074	60	0142	0096	
0096	10	0000	0134	
0134	21	0142	0146	$r_i + n$
0146	19	0120	0035	
0035	20	0139	0143	$r_i(r_i + n)$
0143	65	0039	0110	
0110	64	0139	0094	
0094	60	8002	0104	
0104	19	0116	0062	
0062	31	0008	0083	U_i
0083.	45	0086	0091	Is $U_i = 0$?
0086	20	0116	0070	
0070	15	0060	0125	
0125	20	0060	0046	ΣU_i stored in 0060
0091	60	0058	0114	
0114	19	0060	0092	
0092	31	0008	0113	f(x) is left in L. Acc.
0113	69	0117	0132	
0132	24	0119	0124	Restore 0119 to use Δr = 1.
				Enter routine for floating decimal
				answer.
0093	00	0000	0001	
0006	00	0000	0002	Constants
0033	00	0000	000n	
0039	(00	0000	0000)	$\pm (\frac{x}{2})^2$ or $\pm x^2$

PROBLEM: Special Steps for Sin x, Cos x

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0001	67	0004	0009	
0009	16	0012	0017	$\lvert x \rvert - 2\pi$.
0017	46	0020	0009	If result is negative,
0020	15	0012	0067	add 2π.
0067	10	0004	0076	
0076	46	0088	0059	
0059	20	0013	0066	Store reduced argument with sign.
0088	11	0004	0141	
0141	66	8002	0059	
0066	69	0019	0022	
0022	24	0021	0002	Set 0021 to skip divide x by 2.
0002	69	0005	0008	
0008	24	0106	0003	Set 0106 to skip A(x) routine.
0003	69	0108	0061	
0061	24	0119	0030	Set 0119 to use $\Delta r = 2$.
0030	65	0033	0037	
0037	46	0049	0050	
0050	69	0013	0055	Use A(x) = x for sin x
0055	24	0058	0021	
0049	69	0052	0055	Use A(x) = 1 for cos x.
0021	65	0013	0118	
0118	20	0023	0026	Place x in x ÷ 2 storage.
0026	60	8001	0082	
0082	19	8001	0102	x^2
0102	35	0002	0069	
0069	(66)	8003	0034	
0034	20	0039	0100	Store $-x^2$
0100	69	0103	0106	
0106	24	0069	(0122)	
0122	69	0075	0078	
0078	24	0021	0073	Restore 0021 to divide x by 2.
0073	69	0080	0084	Restore 0106 for A(x) routine;
0084	24	0106	0111	enter g(x) routine.
0019	65	0013	0118	
0005	24	0069	0122	
0108	60	0006	0147	
0075	65	0013	0068	Constants
0080	24	0069	0072	
0012	06	2831	8531	
0052	01	0000	0000	
0033	00	0000	000n	
0004	(00	0000	0000)	Argument

PROBLEM: Alterations for Sinh x, Cosh x

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0011	69	0014	0018	
0018	24	0069	0016	Set 0069 to use + x^2
0016	69	0004	0007	
0007	24	0013	0066	Place unreduced x in 0013.
0066	69	0019	0022	
0022	24	0021	0002	Set 0021 to skip divide by 2.
				Continue in sin x, cos x routine.
0014	65	8003	0034	
0019	65	0013	0118	Constants
0004	(00	0000	0000)	Argument

PROBLEM: Converting f(x) to Floating Decimal

LOCATION	OP.	DATA	INSTRUCTION	REMARKS
0124	20	0130	0133	Store fixed decimal answer.
0133	60	8002	0144	
0144	36	0000	0121	Shift Left and count.
0121	47	0126	0126	Br. Overflow if f(x) = 0.
0126	21	0131	0085	Store shifted answer.
0085	46	0140	0042	Is answer negative?
0042	61	8002	0148	
0148	10	0115	0027	Positive: Add (51 - count)
0027	15	0131	0137	to shifted answer.
0137	31	0002	0149	
0149	60	8002	0077	
0077	15	0130	0123	Exit to main routine.
0140	61	8002	0053	Negative: Subtract (51 + count)
0053	11	0115	0027	from shifted answer.
0115	00	0000	0051	Constant
0130	(00	0000	0000)	Temporary Storage
0131	(00	0000	0000)	" "