DP-1168

AEC RESEARCH AND DEVELOPMENT REPORT

QUANTITATIVE ANALYSIS OF REACTOR SAFETY

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Printed in the United States of America
Available from
Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

663684

DP-1168

Mathematics and Computers
(TID-4500)

QUANTITATIVE ANALYSIS OF REACTOR SAFETY

by

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November 1968

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CONTRACT AT(07-2)-1 WITH THE
UNITED STATES ATOMIC ENERGY COMMISSION

ABSTRACT

The behavior of the safety mechanisms of a nuclear reactor is modeled, and the probabilities of each incident whose consequences are of safety concern are calculated by either of two computer programs. The model accounts for scheduled inspection and maintenance of all mechanical devices.

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INTRODUCTION

The risks of operating nuclear reactors are difficult to assess because, fortunately, there is little experience with the incidents whose consequences are of major concern. However, the methods known generally as reliability analysis can be applied to quantify these risks and thus to remove some of the subjectivity often associated with the assessment. The objective of this report is to propose models that represent the safety features of nuclear reactors and to develop mathematical methods for a quantitative analysis of the safety models of the reactors. It is inherent in these methods that the answers are always stated as probabilities, very few of which are found to be zero with complete assurance. In spite of these inherent uncertainties in the final results, good models and good analyses will provide the proper relative rankings of the risks. Also, a major benefit of these methods is the pin-pointing of the weaknesses in a given system and the indication of those areas in which a given effort will produce the greatest improvement in safety.

SUMMARY

A general model was developed to represent the safety aspects of the behavior of a nuclear reactor, and two computer programs were written to calculate the probabilities of incidents of concern during reactor operation. These incidents, generally, have a Poisson distribution so that each can be characterized by a mean time between occurrences. This feature in conjunction with a measure of the consequences of the incident provides a relative measure of the risks associated with each postulated incident. This form of safety analysis indicates the sources of the greatest risks and the areas in which improvement programs will be most effective.

In the model, each incident whose consequences are of interest is assumed to be preceded by a sequence of events starting at some initiating event. At each member in the postulated sequence there may be functions, such as safety trips and personnel operations, that effectively stop the sequence prior to any incident of significant consequences. Thus, any incident in a sequence is the result of an initiating event and the failure of all arresting functions at intervening steps. When mechanical devices fail, they are assumed to remain inoperative until repaired. The model accounts for scheduled inspection and maintenance of all equipment in the system.

The computer program, INTEG, calculates the probabilities of each sequence of events under the assumption that the failure of each of the safety mechanisms in that sequence is independent of all others. The advantage of this program is its capability to analyze large systems, viz. 20-50 members in the sequence with safety devices at each member.

The computer program, INSPEC produces the same results as INTEG but is limited to systems with no more than seven safety devices. However, INSPEC can analyze these smaller systems or subsystems without the limitation of independence of individual events. Thus, the effects of any desired intercorrelation between failures of safety mechanisms and personnel actions may be incorporated into the calculations. This program is based on the simulation of reactor operation as a Markov process.

Formulas and tables are provided to estimate the frequencies of rare events at several confidence levels.

DISCUSSION

BASIC ASSUMPTIONS

The first step in the quantitative analysis of the safety of a nuclear reactor is the formulation of a mathematical model that accounts for all the pertinent features and parameters of the real system. The model must describe adequately the behavior of the system.

The model that is required for an analysis of safety is one that describes the abnormal behavior of the reactor because safety is always assured as long as the reactor follows expected patterns. The safety of the reactor is dependent upon the operation of corrective action systems such as safety systems, procedures, and personnel actions to counteract potential perturbations that would produce deviations from the expected behavior. Thus, the model that is applicable in safety analysis is the one that represents the interaction between the corrective action systems and the potential perturbations.

The model must necessarily represent a stochastic process, i.e., a process by which a system assumes alternative conditions throughout time in accordance with probabilistic laws. Random walk processes of the following type are often cited to illustrate stochastic processes. At time zero, a counter is positioned at the origin on a linear axis. At each unit of time, a fair coin is tossed and if heads appears the counter is moved one unit to the right and if tails appears the counter is moved one unit to the left. Whenever the counter reaches a position three units either to the right or to the left of the origin, it thereafter remains fixed. Thus, the position of the counter is altered throughout time in accordance with probabilistic laws and the process of alteration is a stochastic process.

With only this description of the process, the position of the counter after n time units cannot be specified with certainty. However, the probability of the counter being in a specified position after n units of time can be calculated. Certain general characteristics of the problem can be recognized. The sum of the probabilities of finding the counter in each of the seven alternative positions is invariant in time and is always equal to unity. Sooner or later the counter can reasonably be expected to end at one of the extreme positions and must necessarily remain fixed thereafter. Thus, the probability of each extreme position

approaches 1/2 as n approaches infinity. If an experiment were conducted in accordance with the above description, the number of time units for the counter to reach an extreme position could be counted. Conceptually, there is a mean time to go from the origin to an extreme position and it is merely the arithmetic average of the times for all possible experiments. In a reactor system, the mean time to the occurrence of an event of concern is a convenient quantitative statement of safety.

GENERAL MODEL

The model postulated here to represent the reactor system is shown schematically in Figure 1. Events are shown in boxes and labeled E_i. Events are divided into two categories, initiating and complex. Those which have no connections from below; e.g., E_1, E_2, E_3, E_4, and E_7 are initiating events. All other events are complex. An initiating event, E_i, is assumed to be a random event having a Poisson distribution in time and characterized by an average frequency λ_i. The time intervals between occurrences of events have an exponential distribution characterized by a mean time, $1/\lambda_i$, between occurrences. Complex events are the result of the occurrence of an initiating event plus the existence of certain qualifying conditions shown in the diagram as Q_{ij}. In a reactor, these qualifying conditions are the nonfunctioning of safety circuits and procedural actions. For example, the neutron flux will exceed the prescribed limits, if a control rod drives out and none of the High Level Flux Monitors function. Thus, in the diagram, events and a set of qualifying conditions are always connected by an "and" symbol ⌓ which leads to a complex event. The "or" symbol ⊕ is used to permit the representation of alternate sequences of events.

The essentials of a diagram such as that shown in Figure 1 always exist in some explicit or implicit form during the design of a reactor's safety system. There is always a set of postulated initiating events because these are just the occurrences for which safety circuits are designed. The presumption that certain safety circuits may fail leads to the recognition that subsequent events in the sequence may occur and that additional safety circuits are desirable to terminate the sequence at the earliest possible member.

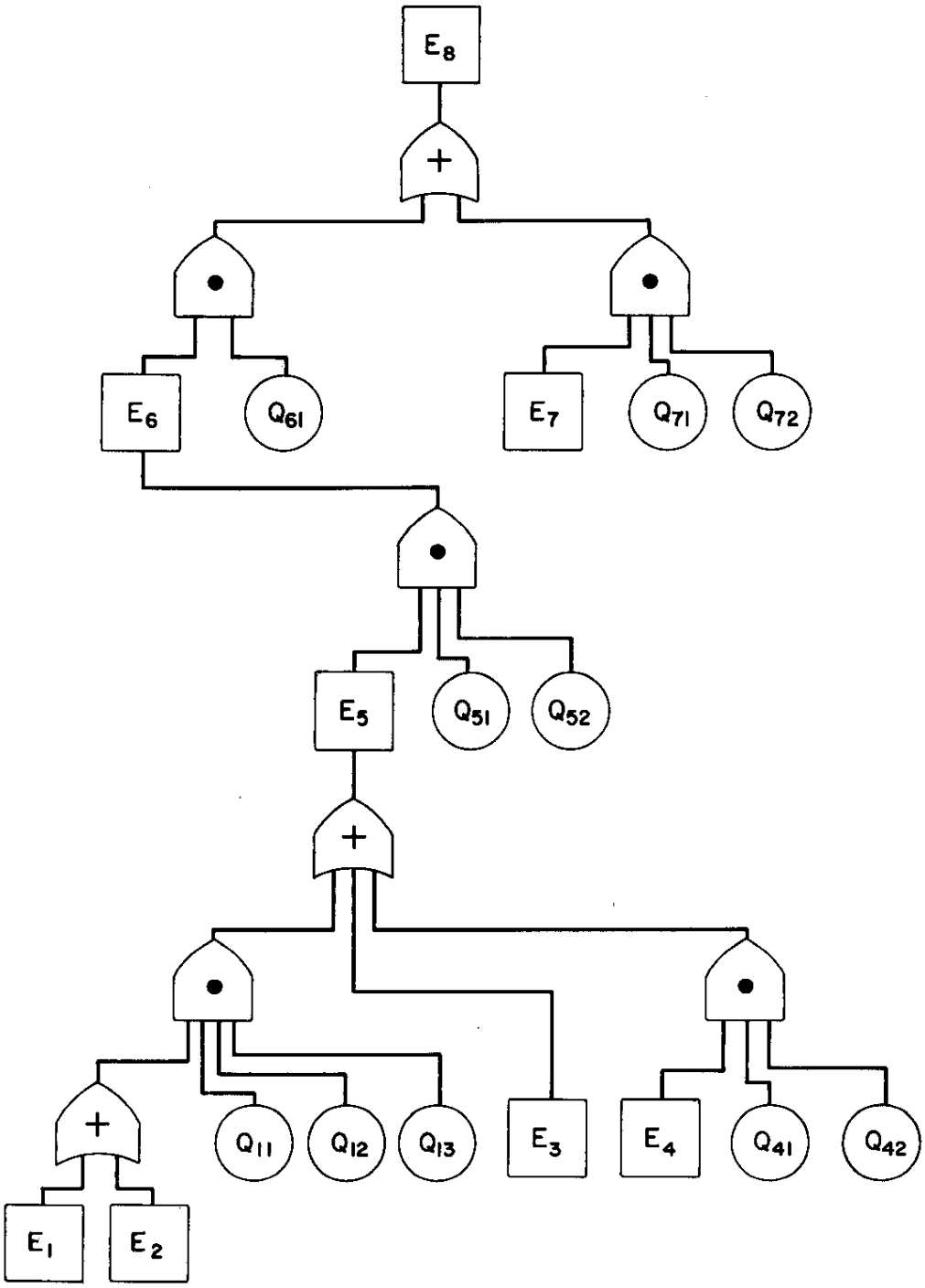


FIG. 1 LOGIC DIAGRAM FOR SEQUENCE OF EVENTS

MATHEMATICAL ANALYSIS – I

Two alternative methods of quantitative evaluation of reactor safety are presented. The first has the capability of treating large systems of the form shown in Figure 1 but with certain restrictions on the postulates applicable to the qualifying conditions. The second method can treat systems having several qualifying conditions and can deal with a variety of assumptions concerning the time behavior of the qualifying conditions.

The first method is designated the "integration method" and the applicable computer program is called INTEG because the major portion of the computer program is a numerical integration routine. As mentioned above, the initiating events, E_i, are assumed to be random events with average frequencies, λ_i. Any qualifying condition, Q_{ij}, is considered to exist from the time of occurrence of a failure in that qualifier until it is repaired and returned to operability. The failure of a qualifier is also assumed to be a random event having a Poisson distribution with a failure frequency of λ_{ij}. A specified complex event initiated by a specified initiating event occurs whenever the initiating event occurs at a time when all the applicable qualifying conditions exist. The qualifiers are assumed to be subject to an inspection and maintenance procedure that prevents the otherwise gradual but certain deterioration of the qualifier system.

The probabilistic analysis of the time behavior of such a system is considered in the following manner. Time, beginning at an arbitrary epoch of concern, is considered to be divided into intervals whose end points are the epochs at which one or more qualifiers are inspected, maintained, and known to be in the operative condition. During each time interval, the behavior of the system is completely probabilistic but at the end of each interval the system is subjected to a deterministic modification. Thus, each time interval may be considered an independent trial in the same sense that the successive tosses of four coins would be independent trials in the determination of the probability of the appearance of four heads.

The probability of a specified complex event during a time from zero to T_n where

$$T_n = \sum_{i=1}^{n} \Delta T_i$$

is given by

$$P_\epsilon(T_n) = 1 - \prod_{i=1}^{n} [1 - P_\epsilon(\Delta T_i)] \qquad (1)$$

where

$P_\epsilon(\Delta T_i)$ = probability of the complex event during the i^{th} time interval

The expression $1 - P_\epsilon(\Delta T_i)$ is the probability that the complex event did not occur during ΔT_i, and the product of the probabilities of all these independent trials is the probability that the complex event did not occur in any ΔT_i. Consequently, unity minus this product is the probability that the complex event occurred at least once during the interval zero to T_n.

The bulk of the effort in evaluating equation (1) is the calculation of all of the $P_\epsilon(\Delta T_i)$. The basic equation that is applicable to the model is

$$\frac{dP_\epsilon(t)}{dt} = [Q(t) - P_\epsilon(t)]\ \lambda_x \qquad (2)$$

where

$Q(t)$ = probability that the qualifying conditions exist at time t, including the probability that the complex event has also occurred by time t

$P_\epsilon(t)$ = probability that the complex event has occurred during time zero to t

and λ_x = frequency of the initiating event

Both $Q(t)$ and $P_\epsilon(t)$ are cumulative distribution functions and, thus, they are monotonically increasing with time and have values between zero and unity. The incremental increase, $\frac{dP_\epsilon(t)}{dt}$, in $P_\epsilon(t)$ between t and t + dt is just the probability that the complex event happens for the first time during t to t + dt. Because $Q(t)$ includes the probability, $P_\epsilon(t)$, that the complex event has already occurred, the probability, $P_\epsilon(t)$, must be subtracted from $Q(t)$ before being multiplied by λ_x to obtain the incremental increase in $P_\epsilon(t)$.

The applicability of equation (2) is illustrated in a simple case in the following example. What is the probability that in n successive rolls of a die a six will appear after an ace? In analogy to the reactor model, the appearance of an ace produces the qualifying condition and the appearance of a six is the initiating event. In this example, time corresponds to successive rolls of the die and so increases by discrete steps rather than

continuously. The form of equation (2) that is applicable to this example is

$$\Delta P_\epsilon(n) = [Q(n-1) - P_\epsilon(n-1)]\, p_x$$

The probability, p_x, of the initiating event is just the probability of a six on any roll and thus is a constant equal to 1/6. $Q(n-1)$ is the probability of an ace on some roll prior to the n^{th} and is equal to $1 - (5/6)^{n-1}$. From these numerical values, the successive values of $P_\epsilon(n)$ can be constructed as shown in the following tabular form.

n	p_x	$Q(n-1)$	$P_\epsilon(n-1)$	$\Delta P_\epsilon(n)$	$P_\epsilon(n)$
1	$\frac{1}{6}$	0	0	0	0
2	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{36}$	$\frac{1}{36}$
3	$\frac{1}{6}$	$\frac{11}{36}$	$\frac{1}{36}$	$\frac{10}{216}$	$\frac{16}{216}$
4	$\frac{1}{6}$	$\frac{91}{216}$	$\frac{16}{216}$	$\frac{75}{1296}$	$\frac{171}{1296}$

In each instance, $P_\epsilon(n)$ is the probability that a six will follow an ace at least once in n successive rolls and $\Delta P_\epsilon(n)$ is the probability that a six follows an ace for the first time on the n^{th} roll.

The differential equation (2) can be converted to an integral equation, which is the form utilized in the computer program. The equation

$$\frac{dP_\epsilon(t)}{dt} = [Q(t) - P_\epsilon(t)]\, \lambda_x$$

is a first order, linear differential equation and thus can be integrated by the usual methods to give

$$P_\epsilon(T) = \int_0^T Q(t)\, \lambda_x dt\, e^{-\lambda_x(T-t)} \qquad (3)$$

For a probability to be equal to an integral, or a summation in a discrete case, the integrand must be the probability of a member of a set of mutually exclusive and exhaustive outcomes covering the domain of the integration. Therefore, an examination of the integrand is instructive for the comprehension of the computational scheme. $Q(t)$ is the probability that the qualifying conditions exist at time t, and $\lambda_x dt$ is the probability that the initiating event occurs between t and t + dt. Thus, the product of these factors is the probability that the complex event occurs during t to t + dt. The factor $e^{-\lambda_x(T-t)}$ is the probability that the initiating event does not occur again between t and T. This last probability is necessary to make the set of outcomes mutually exclusive because otherwise multiple occurrences of the initiating event after the qualifying conditions exist would be counted multiple times when in fact the complex event has occurred only once. The scheme of equation (3) postpones the counting until the last occurrence of the initiating event during the time interval zero to T.

An extension of the earlier example of dice rolling in terms corresponding to equation (3) is

$$P_\epsilon(4) = \sum_{i=1}^{4} \left[1 - \left(\frac{5}{6}\right)^{i-1}\right] \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{4-i}$$

$$= 0 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3 + \frac{1}{6} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 + \frac{11}{36} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{91}{216} \cdot \frac{1}{6} \cdot 1$$

$$= \frac{171}{1296}$$

which is identical with the result in the earlier calculation.

During any interval $\Delta T_i = T_i - T_{i-1}$ where T_{i-1} is an epoch at which some qualifier was examined and found to be operative so that $Q(T_{i-1}) = 0$

$$P_\epsilon(\Delta T_i) = \int_{T_{i-1}}^{T_i} Q(t)\ \lambda_x dt\ e^{-\lambda_x(T_i - t)}$$

In this fashion, the quantities indicated in equation (1) are calculated.

An explicit expression for Q(t) is required to permit numerical calculations. Q(t) is the probability that all of the applicable qualifiers are inoperative at time t. If the failures of the qualifiers are all independent, the probability that all are inoperative is the product of the probabilities of individual failures. For random failures with Poisson distributions, the probability of inoperability is

$$1 - e^{-\lambda_i(t - \tau_{ik})}$$

where

λ_i = average frequency of failure

t = time

and

τ_{ik} = time of previous inspection when the qualifier was known to be operative

Thus,

$$Q(t) = \prod_{i=1}^{m} \left[1 - e^{-\lambda_i(t - \tau_{ik})}\right] \tag{4}$$

When all of these factors are combined in the appropriate manner, equation (1) is

$$P_\epsilon(T_n) = 1 - \prod_{i=1}^{n} \left\{1 - \int_{T_{i-1}}^{T_i} \prod_{j=1}^{m} \left[1 - e^{-\lambda_j(t - \tau_{jk})}\right] \lambda_x dt \left[e^{-\lambda_x(T_i - t)}\right]\right\} \tag{5}$$

The computer program INTEG calculates the value of equation (5) as a function of T_n. The required input is

λ_x = frequency of the initiating event

λ_j = frequency of failure for the j^{th} qualifier

$\Delta\tau_j$ = interval between inspections of the j^{th} qualifier

and

τ_{jo} = time of the initial inspection of the j^{th} qualifier

The τ_{jk} are then given by

$$\tau_{jk} = \tau_{jo} + k\Delta\tau_j, \qquad k = o, 1, 2, 3, \ldots$$

In the computer program, all of the τ_{jk} are arranged in order by magnitude and to each distinct value in the set is assigned a T_i, also in order of magnitude. The indicated integration is performed by summing rectangles whose heights are the values of the integrand at the midpoints of the intervals. The number of intervals per unit of time is specified in the program input. The program output includes T_n, $P_\epsilon(T_n)$, and $-\ell n[1 - P_\epsilon(T_n)]/T_n$, for selected values of T_n up to a designated maximum T_n. The program is explained in more detail in Appendix A.

The cumulative distribution function $P_\epsilon(T_n)$ is of a complicated form not included among the simpler forms covered in texts on probability. The function over limited ranges such as the times between inspections of qualifiers is a modified gamma type. However, when $P_\epsilon(T_n)$ is considered over long periods of time, its characteristics are similar to that of a simple exponential distribution. This is demonstrated in numerical calculations by the fact that $-\ell n[1 - P_\epsilon(T_n)]/T_n$ approaches a constant as T_n becomes large. This constant, λ_ϵ, is the average frequency of the complex event and $1/\lambda_\epsilon$ is the mean time to the occurrence of the complex event. These relationships are shown by setting

$$-\ell n[1 - P_\epsilon(T_n)]/T_n = \lambda_\epsilon$$

and rearranging to the readily recognized form of the exponential distribution,

$$P_\epsilon(T_n) = 1 - e^{-\lambda_\epsilon T_n} \qquad (6)$$

Table I is an illustration of the application of the computer program to an example having the exact form of Figure 1. The input data are shown in Table II.

An examination of the results of Table I shows that event E_8 has nearly as high a frequency as E_6 and thus that the safety circuit or qualifier Q_{81} is not very effective. If E_8 is of appreciably greater consequences than E_6, then the system has a decided weakness at Q_{81}. Otherwise, the system is rather well balanced with no initiating event contributing excessively to any total frequency. Although E_3 is shown as an initiating event, there are no qualifiers between it and E_5 so that in essence E_3 is really the spontaneous occurrence of E_5. If this is truly the interpretation intended in the model, then the contribution from E_3 can be decreased only by improving the safety circuits on E_5 and E_6. Modifications at these levels will improve the overall safety of the system but does not lessen the fractional contribution of E_3 as compared to E_1, E_2, and E_4.

TABLE I

Example of Reactor Safety Calculation by Computer Program INTEG

Event	Frequency Resulting from Initiating Event					Total Frequency
	E_1	E_2	E_3	E_4	E_7	
E_1	1.0×10^{-2}	-	-	-	-	1.00×10^{-2}
E_2	-	1.2×10^{-2}	-	-	-	1.20×10^{-2}
E_3	-	-	4.0×10^{-4}	-	-	4.00×10^{-4}
E_4	-	-	-	5.0×10^{-3}	-	5.00×10^{-3}
E_5	6.80×10^{-5}	7.99×10^{-5}	4.00×10^{-4}	7.82×10^{-5}		6.26×10^{-4}
E_6	8.56×10^{-6}	1.01×10^{-5}	4.78×10^{-5}	9.98×10^{-6}	-	7.64×10^{-5}
E_7	-	-	-	-	1.00×10^{-4}	1.00×10^{-4}
E_8	5.00×10^{-6}	5.91×10^{-6}	2.70×10^{-5}	5.97×10^{-6}	5.05×10^{-8}	4.39×10^{-5}

TABLE II

Input Data for Problem of Table I

Initiating Event	Frequency λ_j
E_1	1.0×10^{-2}
E_2	1.2×10^{-2}
E_3	4.0×10^{-4}
E_4	5.0×10^{-3}
E_7	1.0×10^{-4}

Qualifier	Frequency of Failure λ_{ij}	Interval Between Inspections $\Delta\tau_{ij}$	Time of First Inspection τ_{io}
Q_{11}	5×10^{-3}	100	0
Q_{12}	5×10^{-3}	100	75
Q_{13}	5×10^{-3}	100	50
Q_{41}	4×10^{-3}	75	0
Q_{42}	4×10^{-3}	75	50
Q_{51}	6×10^{-3}	150	75
Q_{52}	6×10^{-3}	150	100
Q_{61}	1×10^{-2}	200	0
Q_{71}	1×10^{-3}	50	0
Q_{72}	1×10^{-3}	50	25

The implication of equation (6) that the complex events have an average behavior in time that can be described by a simple exponential distribution is illustrated in Figure 2 where the probability of event 8 initiated by event 2, as described in Figure 1, is plotted as a function of time. Individual points do deviate from the straight line relationship expressed in equation (6), but the deviations do not increase with time so that the long-term behavior is adequately expressed by the equation (6). The pattern in this instance repeats each 600 time units because this is the least common multiple of all the intervals between inspections of the applicable qualifiers, viz., 100, 150, and 200.

In the illustrative example, the characteristics of the qualifiers have been stated in the explicit quantitative parameters of failure rate and maintenance schedule. If in some cases, the qualifiers cannot be described in these terms, then their behavior is incorporated in the model in a modified form. For example, an operator is not adequately represented as a mechanism with a failure rate and a maintenance schedule. However, a probability that an operator will fail to take the prescribed actions should

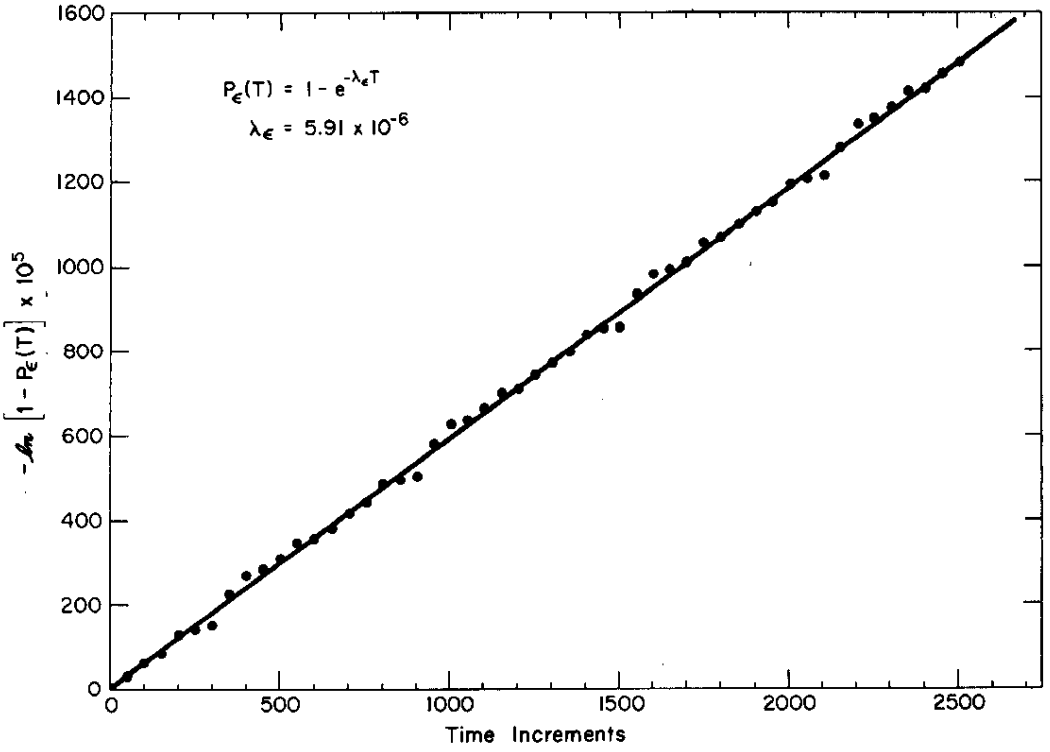


FIG. 2 PROBABILITY OF COMPLEX EVENT AS A FUNCTION OF TIME
(Data from Table II for event 8 initiated by event 2)

be incorporated into the analysis. This feature can be included in the formal framework of the model by including a term for the operator qualifier in the usual form $(1 - e^{-\lambda t})$. An appropriate λ and T are chosen so that

$$\frac{\int_0^T (1 - e^{-\lambda t}) dT}{\int_0^T dt} = f \qquad (7)$$

where f equals the fraction of the time that the operator fails. When the integration indicated in equation (7) is performed,

$$\frac{1}{\lambda T} (1 - e^{-\lambda T}) = 1 - f \qquad (8)$$

For example, if $\lambda T = 0.5$,

$$f = 0.213$$

or the qualifying condition would exist about 21% of the time on the average. Inclusion of the qualifier in this form will not produce exactly the 21% factor except when $T \rightarrow 0$. Thus, in general, T should be taken as small as possible without unduly lengthening the computation.

There may be some events whose qualifiers cannot be described in the specific forms implied in the above description of the model but yet judgment would indicate the existence of some qualification. For example, the melting of a fuel element might be expected to terminate at some stated fraction without the operation of specific mechanical devices. To account for such circumstances, the successive fractional meltings can be described as a sequence of events and the desired qualifications incorporated in the computation in the form of equation (7).

MATHEMATICAL ANALYSIS – II

The second method of analysis is based upon the representation of the time behavior of the model as a Markov process and employs the computer program, INSPEC. This method does not require the independence of the failures of the qualifiers and does not require the recovery of the operability of a qualifier to occur at a scheduled time. This latter feature permits the representation of qualifiers, such as personnel actions, to be a purely random

condition, alterable at each unit of time. However, this more elaborate representation is limited to systems or subsystems that contain no more than seven qualifiers. This limitation is purely computational. The solution by this method requires multiplication of $(2^n + 1)$ by $(2^n + 1)$ matrices, where n is the number of qualifiers. Thus, the system that contains seven qualifiers involves matrices that are 129 x 129. Larger problems involve matrices that are inconveniently large.

The example, cited earlier, of a stochastic process in which a counter moves from an original position at the origin in successive steps with probability 1/2 to positions between -3 and +3 is also a Markov process. The Markov property is ascribed to the process because the position of the counter after a transition is dependent only on the position immediately prior to the transition. For example, if the counter were known to be in position (+1) then after the next transition the counter has probabilities of 1/2 of being in either of the positions, 0 and +2. Knowledge that the counter was in position (-2) five transitions earlier is completely irrelevant because such knowledge adds nothing to what is known when the immediately prior position is specified.

The complete description of this Markov process is given by the following set of equations, one for each of the seven possible positions that are generally designated as states.

$$
\begin{aligned}
\pi_{-3}(n+1) &= \tfrac{1}{2}\pi_{-2}(n) + \pi_{-3}(n) \\
\pi_{-2}(n+1) &= \tfrac{1}{2}\pi_{-1}(n) \\
\pi_{-1}(n+1) &= \tfrac{1}{2}\pi_{0}(n) + \tfrac{1}{2}\pi_{-2}(n) \\
\pi_{0}(n+1) &= \tfrac{1}{2}\pi_{-1}(n) + \tfrac{1}{2}\pi_{+1}(n) \\
\pi_{+1}(n+1) &= \tfrac{1}{2}\pi_{0}(n) + \tfrac{1}{2}\pi_{+2}(n) \\
\pi_{+2}(n+1) &= \tfrac{1}{2}\pi_{+1}(n) \\
\pi_{+3}(n+1) &= \tfrac{1}{2}\pi_{+2}(n) + \pi_{+3}(n)
\end{aligned}
\qquad (9)
$$

where $\pi_j(i)$ is the probability of the j^{th} state after the i^{th} transition. When these seven equations are added together, the sum of the probabilities for the seven states before a transition

equals the sum after a transition. This is an obvious characteristic of any system because the probabilities of all possible states must total to unity. Customarily, the set of equations is written as one matrix equation as follows:

$$\pi(n + 1) = \pi(n) \cdot P \qquad (10)$$

where π without a subscript stands for the row vector,

$$\pi(i) = [\pi_{-3}(i),\ \pi_{-2}(i),\ \pi_{-1}(i),\ \pi_{0}(i),\ \pi_{+1}(i),\ \pi_{+2}(i),\ \pi_{+3}(i)]$$

and P stands for the matrix,

$$P = \begin{array}{c} \text{State} \\ -3 \\ -2 \\ -1 \\ 0 \\ +1 \\ +2 \\ +3 \end{array} \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & +1 & +2 & +3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix, P, is termed a stochastic transition matrix. Since all of its elements are constants that do not change with time, the whole matrix is a constant. The individual elements are the conditional probabilities of specific transitions. They are each the probability of a transition to the state corresponding to the i^{th} column given that the system is in the state corresponding to the j^{th} row. For example, given that the counter is at position (+2) corresponding to the sixth row, the probability of a transition to position (+1) corresponding to the fifth column equals 1/2. The sum of the elements in any row must equal unity because given that the system is in a specified state, the system must with certainty be in some state after the transition.

The variable part of the problem is contained in the probability vector, $\pi(i)$, each of whose elements represents the probability of a specified state after the i^{th} transition. The sum of the elements of this vector always equals unity because the system must with certainty be in some state.

All that is necessary to determine the dynamic behavior of the system is to write down the P matrix and specify a starting value for the $\pi(i)$ vector. Oftentimes the starting condition of the system is presumed known exactly and some one state is assigned a probability of unity. In the above example, the counter started at the origin so the probability vector is

$$\pi(0) = [0, 0, 0, 1, 0, 0, 0]$$

The solution is then carried out by successive multiplications of the probability vector by the P matrix.

$$\begin{aligned} \pi(1) &= \pi(0) \cdot P \\ \pi(2) &= \pi(1) \cdot P \\ \pi(3) &= \pi(2) \cdot P \\ \pi(n+1) &= \pi(n) \cdot P \end{aligned}$$

Also, successive substitutions in the previous set of equations yield

$$\pi(n) = \pi(0) \cdot P^n \qquad (11)$$

The following numerical results illustrate the results of carrying out the solution procedure on the illustrative example.

	Probabilities of States						
Transition	π_{-3}	π_{-2}	π_{-1}	π_{0}	π_{+1}	π_{+2}	π_{+3}
0	0	0	0	1	0	0	0
1	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
2	0	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$	0
3	$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$
4	$\frac{1}{8}$	$\frac{3}{16}$	0	$\frac{3}{8}$	0	$\frac{3}{16}$	$\frac{1}{8}$
5	$\frac{7}{32}$	0	$\frac{9}{32}$	0	$\frac{9}{32}$	0	$\frac{7}{32}$
6	$\frac{7}{32}$	$\frac{9}{64}$	0	$\frac{9}{32}$	0	$\frac{9}{64}$	$\frac{7}{32}$
7	$\frac{37}{128}$	0	$\frac{27}{128}$	0	$\frac{27}{128}$	0	$\frac{37}{128}$
8	$\frac{37}{128}$	$\frac{27}{256}$	0	$\frac{27}{128}$	0	$\frac{27}{256}$	$\frac{37}{128}$
9	$\frac{175}{512}$	0	$\frac{81}{512}$	0	$\frac{81}{512}$	0	$\frac{175}{512}$

The general characteristics referred to above are apparent in the table. The sums of the rows are always equal to unity. Also, the probabilities, π_{-3} and π_{+3}, are monotonically increasing and are approaching 1/2.

This example is a finite Markov chain because there is only a finite set of possible states and the transition probabilities are constants. There are two kinds of states in this problem, viz., transient and absorbing. States (-2), (-1), (0), (+1), and (+2) form a transient set. By proper transitions, it is possible to go from one state to any other state in this transient set. However, when a transition out of the transient set takes place, the set is never re-entered. The two states (-3), and (+3) are absorbing states because once they are entered they are never left.

If our only interest in this problem is how long it takes to reach an absorbing state, the numerical work can be reduced considerably. The reduction consists of combining all the states in the transient set into one state so that the result is only a three-state system. Also, because of the fact that states (-3) and (+3) can be entered only on odd-numbered transitions, the number of transitions can be reduced to one half. Thus, the solution is now given by the equation

$$\left[\pi_{-3}(n+1),\ \pi_{0}(n+1),\ \pi_{+3}(n+1)\right] = \left[\pi_{-3}(n),\ \pi_{0}(n),\ \pi_{+3}(n)\right] \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

The numerical tabulation of the results is then,

	Probabilities of States			
Transition - n	π_{-3}	π_{0}	π_{+3}	$(\pi_{-3} + \pi_{+3})$
0	0	1	0	0
1	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{4}$
2	$\frac{7}{32}$	$\frac{18}{32}$	$\frac{7}{32}$	$\frac{14}{32}$
3	$\frac{37}{128}$	$\frac{54}{128}$	$\frac{37}{128}$	$\frac{74}{128}$
4	$\frac{175}{512}$	$\frac{162}{512}$	$\frac{175}{512}$	$\frac{350}{512}$

The number of transitions taken to go from the origin to an absorbing state in any particular instance is a random variable. A random variable is a function, in the mathematical sense, whose domain is all the possible outcomes of an experiment subject to probabilistic fluctuations and whose range is the real numbers. In this example, the outcome of each experiment in which the counter starts at the origin and eventually ends in an absorbing state defines a real number which is the number of transitions required in the experiment. Here, the range of the random variable is the set of all positive integers.

Random variables are always characterized by cumulative distribution functions and density functions. The last column headed $(\pi_{-3} + \pi_{+3})$ in the above table is the cumulative distribution function of the random variable described above. The cumulative distribution function is merely the probability that the random variable will be equal to or less than a stated value. This is expressed as

$$F(n) = \Pr\ (N \leqslant n) \tag{12}$$

The density function of a random variable is the probability that the random variable will have a particular value. This is expressed as

$$f(n) = \Pr\ (N = n) \tag{13}$$

The relationship between the cumulative distribution function and the density function is

$$F(n) = \sum_{i=1}^{n} f(i) \tag{14}$$

When the random variable is a continuous function, the summation is an integral and in general the range is assumed to be from minus to plus infinity. Thus,

$$F(x) = \int_{-\infty}^{x} f(y)\ dy \tag{15}$$

For the continuous case, $f(y)$ is the probability that the value of the random variable will fall between y and $y + dy$.

The expected value, also known as the mean or average value of a random variable, is given by

$$E(X) = \int_{-\infty}^{+\infty} y\ f(y)\ dy \tag{16}$$

This is exactly equivalent to the common concept of arithmetic averaging. Each possible value y of the random variable X is weighted in accordance with its probability and summed.

The density function for the random variable described above can be constructed as in the following table.

Transition - n	F(n)	f(n)
0	0	-
3	$\frac{1}{4}$	$\frac{1}{4}$
5	$\frac{14}{32}$	$\frac{3}{16}$
7	$\frac{74}{128}$	$\frac{9}{64}$
9	$\frac{350}{512}$	$\frac{27}{256}$

The true number of transitions are listed in this table rather than that in the condensed system given in the preceding table. By induction,

$$f(n) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{\frac{n-3}{2}}$$

where

$$n = 3, 5, 7, \cdots$$

The sum of all values of f(n) for all permissible values of n is equal to unity as expected since the summation is over all possible values of the random variable. Then the expected or mean value of the random variable, N, is

$$E(N) = \sum_n n \, \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{\frac{n-3}{2}}$$

where $n = 3, 5, 7, \cdots$

The series is

$$E(N) = 3 \cdot \frac{1}{4} + 5 \cdot \frac{3}{16} + 7 \cdot \frac{9}{64} + \cdots$$

$$= 9$$

Thus, on the average, the counter starting at the origin will be in one of the absorbing states at the end of nine transitions.

This density function has a property that is quite common to density functions of random variables that are times to occurrence of events that are said to occur at random. It has a maximum at the smallest value of n and decreases by a constant factor for each unit increase in n. Thus, the successive values f(3), f(5), ... form a geometric series. If the random variable were a continuous function, say of time, then the same property would appear as an exponential density function

$$f(t) = \lambda e^{-\lambda t} \tag{17}$$

where $\lambda > 0$ and $t \geq 0$. As required

$$\int_0^\infty \lambda e^{-\lambda t}\, dt = 1$$

Also, the cumulative distribution function is

$$F(\tau) = \int_0^\tau \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda \tau} \tag{18}$$

The expected value of the random variable T is

$$E(T) = \int_0^\infty t\, \lambda e^{-\lambda t}\, dt = \frac{1}{\lambda} \tag{19}$$

This evaluation of the expected value counted time from the epoch zero. However, this density function has the property that the expected value of the random variable is $1/\lambda$ regardless of the starting epoch. This is shown by the equation

$$E(T) = \frac{\int_\tau^\infty (t - \tau)\, \lambda e^{-\lambda t}\, dt}{\int_\tau^\infty \lambda e^{-\lambda t}\, dt} = \frac{1}{\lambda} \tag{20}$$

This means that when a process consists of events that are randomly distributed, i.e., they have an exponential density function, the mean time to the occurrence of an event is the same regardless of the epoch at which the concern commences.

The introduction of an additional feature in the example given above illustrates another type of problem that is of interest in later discussion. Suppose that after the first three transitions the system is examined and if the counter is not in one of the absorbing states it is placed in position zero. The transitions then continue in accordance with the probabilistic rules for another five transitions. Again, the counter is placed in position zero if it is not in an absorbing state. The process of inspection and return of the counter to position zero is repeated indefinitely with the pattern 3, 5, 3, 5, 3, 5, etc. The following table illustrates the results of this altered process.

	Probabilities of States							
Transition	π_{-3}	π_{-2}	π_{-1}	π_0	π_1	π_2	π_3	$\pi_{-3} + \pi_3$
0	0	0	0	1				0
1	0	0	$\frac{1}{2}$	0				0
2	0	$\frac{1}{4}$	0	$\frac{1}{2}$				0
3	$\frac{1}{8}$	0	$\frac{3}{8}$	0				$\frac{1}{4}$
Readjust	$\frac{1}{8}$	0	0	$\frac{3}{4}$				$\frac{1}{4}$
4	$\frac{1}{8}$	0	$\frac{3}{8}$	0	Table is symmetrical.			$\frac{1}{4}$
5	$\frac{1}{8}$	$\frac{3}{16}$	0	$\frac{3}{8}$				$\frac{1}{4}$
6	$\frac{7}{32}$	0	$\frac{9}{36}$	0				$\frac{7}{16}$
7	$\frac{7}{32}$	$\frac{9}{64}$	0	$\frac{9}{32}$				$\frac{7}{16}$
8	$\frac{37}{128}$	0	$\frac{27}{128}$	0				$\frac{37}{64}$
Readjust	$\frac{37}{128}$	0	0	$\frac{27}{64}$				$\frac{37}{64}$
9	$\frac{37}{128}$	0	$\frac{27}{128}$	0				$\frac{37}{64}$
10	$\frac{37}{128}$	$\frac{27}{256}$	0	$\frac{27}{128}$				$\frac{37}{64}$
11	$\frac{175}{512}$	0	$\frac{81}{512}$	0				$\frac{175}{256}$

Transition	Probabilities of States							
	π_{-3}	π_{-2}	π_{-1}	π_0	π_1	π_2	π_3	$\pi_{-3} + \pi_3$
Readjust	$\frac{175}{512}$	0	0	$\frac{81}{256}$				$\frac{175}{256}$
12	$\frac{175}{512}$	0	$\frac{81}{512}$	0				$\frac{175}{256}$
13	$\frac{175}{512}$	$\frac{81}{1024}$	0	$\frac{81}{512}$				$\frac{175}{256}$
14	$\frac{781}{2048}$	0	$\frac{243}{2048}$	0				$\frac{781}{1024}$
15	$\frac{781}{2048}$	$\frac{243}{4096}$	0	$\frac{243}{2048}$				$\frac{781}{1024}$
16	$\frac{3367}{8192}$	0	$\frac{729}{8192}$	0				$\frac{3367}{4096}$

The cumulative distribution function and density function of the random variable, i.e., number of transitions to an absorbing state, can be written down as follows:

Transition - n	F(n)	f(n)
0	0	-
3	$\frac{1}{4}$	$\frac{1}{4} = \frac{1}{4}$
6	$\frac{7}{16}$	$\frac{3}{16} = \frac{1}{4} \cdot \frac{3}{4}$
8	$\frac{37}{64}$	$\frac{9}{64} = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2$
11	$\frac{175}{256}$	$\frac{27}{256} = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^3$
14	$\frac{781}{1024}$	$\frac{81}{1024} = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^4$
16	$\frac{3367}{4096}$	$\frac{243}{4096} = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^5$

This f(n) is similar to the density function found in the earlier example except that specific values of f(n) are associated with larger values of n. For example, above $f(11) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^3$ while in the earlier example $f(9) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^3$. This is surely in accordance with expectation because the mean number of transitions to reach an absorbing state must necessarily be longer. In fact, the mean transition to an absorbing state is now 11.03 as compared to 9 previously.

This latter problem can be approached in a somewhat different manner in which the logic is possibly somewhat more straightforward. After the third transition, the assumption is made that the system is not in an absorbing state, and the process is begun again with the probability of state zero as unity. After the eighth transition, five more transitions in addition to the previous three, the assumption is again made that the system is not in an absorbing state and the process is begun once more with the probability of the state zero equal to unity. The quantity that is calculated in each increment of this process is the conditional probability of no absorption during the increment given that there was no absorbing event during a previous increment, viz.,

$$\Pr\left(\overline{E}_i \middle| \overline{E}_{i-1} \cdot \overline{E}_{i-2} \cdots \overline{E}_1\right) \tag{21}$$

where $\overline{E}_i$ represents the nonoccurrence of an absorbing event during the i^{th} increment.

Then, the unconditional probability of no absorbing event during any number of successive increments is given by

$$\Pr\left(\overline{E}_i \cdot \overline{E}_{i-1} \cdots \overline{E}_2 \cdot \overline{E}_1\right) = \Pr\left(\overline{E}_1\right) \cdot \Pr\left(\overline{E}_2 \middle| \overline{E}_1\right) \cdot \Pr\left(\overline{E}_3 \middle| \overline{E}_2 \cdot \overline{E}_1\right) \cdots \Pr\left(\overline{E}_i \middle| \overline{E}_{i-1}\, \overline{E}_{i-2} \cdots \overline{E}_1\right) \tag{22}$$

The advantage of this latter formula is that for this specific example there are only two different quantities on the right-hand side, viz., the probability of no absorption in three transitions and in five transitions, respectively. From the numerical tables, these values are $\frac{3}{4}$ and $\frac{9}{16}$. Thus, the probability of no absorption in k increments of length three and m increments of length five is

$$\Pr(\overline{E}) = \left(\frac{3}{4}\right)^k \cdot \left(\frac{9}{16}\right)^m$$

Finally, the probability of an absorption is unity minus $P(\overline{E})$ or

$$\Pr(E) = 1 - \left(\frac{3}{4}\right)^k \cdot \left(\frac{9}{16}\right)^m$$

From this formula, the cumulative distribution function of number of transitions to an absorption can be calculated in tabular form as follows where $n = 3k + 5m$.

k	m	n	Pr(E)
1	0	3	$\frac{1}{4}$
1	1	8	$\frac{37}{64}$
2	1	11	$\frac{175}{256}$
2	2	16	$\frac{3367}{4096}$

Comparison with the previous table confirms that the results are identical.

When $k = m = j$, the probability of an absorption may be written

$$Pr(E) = 1 - \left(\frac{27}{64}\right)^{j}$$

or since $8j = n$

$$Pr(E) = 1 - \left(\frac{27}{64}\right)^{\frac{n}{8}}$$

This last formula can be transformed to give it the appearance of the cumulative distribution function of a continuous variable, viz.,

$$P(E) = 1 - e^{-\left(\frac{1}{8} \ell n \frac{64}{27}\right) n}$$

This formula obviously is correct whenever n is a multiple of eight, but is only approximate at other values. On the other hand it gives the correct long-term behavior of the system.

This latter process can be generalized quite readily and applied to very complex systems that consist of a series of finite Markov processes separated by epochs at which information is gained by deterministic means such as inspections and maintenance. The conditional probability of the nonoccurrence of an absorption is calculated for each separate Markov process under the assumption that no absorptions have occurred during any previous process. The product of all these conditional probabilities is the unconditional or total probability of no absorptions during the whole series of processes. Unity minus the probability of no absorptions equals the probability of an absorption during the whole series. It is readily apparent that this value is calculated much more easily by subtracting its complement from unity rather than considering the complex situation of one or more absorptions and their possible distributions throughout the series of Markov processes.

If and when the pattern of inspections and maintenance repeats itself, the product of the conditional probabilities repeats and the form of the equation for the probability of an absorption is

$$Pr(E) = 1 - K^{j} \tag{23}$$

where K is a constant and j is the number of periods of the repeating pattern. If the length of the repeating pattern is T, then

$$Pr(E) = 1 - K^{n/T} \tag{24}$$

where n now is the actual number of transitions. This equation may be written

$$Pr(E) = 1 - e^{-\left(\frac{1}{T} \ell n \frac{1}{K}\right) n} \tag{25}$$

The numerical results of the program INSPEC approximate the results of INTEG to the extent that the continuous process can be represented by the discrete Markov process. The discrete process always underestimates the probability of the complex event but the approximation is better the smaller the interval in the discrete representation. The input data for the continuous process as calculated by INTEG are, λ_i, the frequencies of events and thus can have any real positive value. However, in the discrete process, as calculated by INSPEC, the input are probabilities which always lie in the range zero to unity. The relationship between the two is

$$p = 1 - e^{-\lambda_i \Delta t}$$

where Δt is the time interval of the discrete process. The following table shows a comparison of results of the two computational methods. The input data are:

4 qualifiers, each with failure frequency = 0.005

Initiating event with frequency = 0.01

Probability of Complex Event

		INSPEC	
T	INTEG	$\Delta t = 1$	$\Delta t = 0.1$
50	0.239×10^{-3}	0.201×10^{-3}	0.235×10^{-3}
100	0.474×10^{-2}	0.442×10^{-2}	0.471×10^{-2}
200	0.610×10^{-1}	0.596×10^{-1}	0.609×10^{-1}

REACTOR MODEL

The time behavior of a safety system such as shown in Figure 1 may be considered as a Markov process. The condition of the system at any time prior to an initiating event, E_1, may be described by an assignment to a state, S_1, which is defined by a listing of which qualifiers are operative and which are inoperative. There is one additional state in the set, viz., the complex event E, which consists of all qualifiers inoperative and the occurrence of an initiating event. For a system with a set of N qualifiers, there are $(2^N + 1)$ states. The 2^N states represent all combinations of operative and inoperative qualifiers. For example, when N equals two, there are five states as follows, where O = operative and I = inoperative.

	Qualifiers		
State	Q_1	Q_2	Perturbation
1	O	O	
2	I	O	
3	O	I	
4	I	I	
5	I	I	E_1

As seen in the earlier discussion, all that is needed to describe the behavior of this system are the probabilities of the failures of the qualifiers and the frequency of occurrence of the initiating event E_1. The result of the calculations is the mean time to state 5 or equivalently, how often, on the average, state 5 will occur.

Inspections and maintenance of the qualifiers during the operating period provide specific knowledge of the operability of the qualifiers and necessarily alter the probabilities of the several states. For example, in the above illustration an inspection that verified the operability of qualifier Q_1, would imply that the probabilities of states 2, 4, and 5 are zero. The effect of such knowledge at specified epochs is accounted for in the solution by setting the probabilities of the affected states in the probability vector to zero and normalizing all of the remaining probabilities to unity. The following table shows an example of how the state probabilities are adjusted to account for an inspection.

	State Probabilities				
	π_1	π_2	π_3	π_4	π_5
Before Inspection	0.3	0.2	0.2	0.2	0.1
After Inspection of Qualifier Q_1	0.6	0	0.4	0	0

This form of adjustment is justified by Bayes' theorem. After an inspection verifies the operability of one or more qualifiers, the states may be divided into two categories: A, those which are possible and B, those which are impossible or have zero probability. Bayes' theorem may be expressed as

$$\Pr(A_i|C) = \frac{\Pr(A_i) \cdot \Pr(C|A_i)}{\sum_i \Pr(C|A_i) \cdot \Pr(A_i) + \sum_j \Pr(C|B_j) \cdot \Pr(B_j)} \qquad (26)$$

where

$\Pr(A_i)$, $\Pr(B_j)$ = probability of state A_i or B_j prior to knowledge gained from inspection

$\Pr(A_i|C)$ = probability of state A_i given the knowledge gained from inspection

$\Pr(C|A_i)$, $\Pr(C|B_j)$ = probability that the knowledge gained from inspection is true given that the systems were in state A_i or B_j

By the definition of A_i and B_j, $\Pr(C|A_i) = 1$ and $\Pr(C|B_j) = 0$. Thus after each inspection of one or more actions, the elements of the π probability vector are recalculated as

$$\Pr(A_i|C) = \frac{\Pr(A_i)}{\sum_i \Pr(A_i)} \qquad (27)$$

A computer program, INSPEC, has been written to calculate the time behavior of safety systems. The program is described in detail in Appendix B.

The program solves the problem of the mean time to a complex event, E, given the probabilities of failure of the qualifiers, the probability of the initiating event, and the inspection and maintenance schedule for the qualifiers. The first item that the program computes is the description of all $(2^N + 1)$ states. This consists of a listing of all the integers from zero to $(2^N + 1)$ in binary form, where "0" (zero) represents operative and "1" (one) represents inoperative. This list looks, in part, as follows:

	Q_1	Q_2	Q_3	Q_4	...
S_1	0	0	0	0	
S_2	1	0	0	0	
S_3	0	1	0	0	
S_4	1	1	0	0	
S_5	0	0	1	0	
S_6	1	0	1	0	
⋮					

Next, the program makes up the P matrix from all the failure probabilities. This matrix, as initially formed, contains the two assumptions: (1) that all failures are independent and (2) that when a qualifier once fails, it remains failed until its scheduled inspection and maintenance. Additional input data can be specified to modify the P matrix to override either of these two assumptions. Whenever an operator performance is one of the qualifiers, the second assumption is not a good one. An operator might fail to respond satisfactorily in a particular instance and yet without any apparent intervention could be expected to function properly a short time later. Therefore, the P matrix must always contain probabilities for transitions from all states that include ineffective operators to corresponding states in which the operators function satisfactorily. On the other hand, failed mechanical devices are not likely to function properly until inspected and repaired. There may well be cases in which the failures of mechanical devices are not independent. A set of similar devices may all fail simultaneously from a common cause as for example, failure of a common power supply. Also, the failure of one device in a set may be symptomatic of a general deterioration of the system and additional devices may subsequently fail with increased frequency. The computer program makes up the P matrix to incorporate all of the additionally specified requirements.

The π vector of state probabilities is initially set to the value

$$\pi = [1, 0, 0, 0, \ldots]$$

which corresponds to certainty that all qualifiers are operative. The choice of this initial value is not critical because the progressing solution soon takes on the character of a Poisson process. This means that the epochs of the complex event E are randomly distributed in time and the mean time to the occurrence of the complex event, E, is the same regardless of the epoch at which the observation of the process starts.

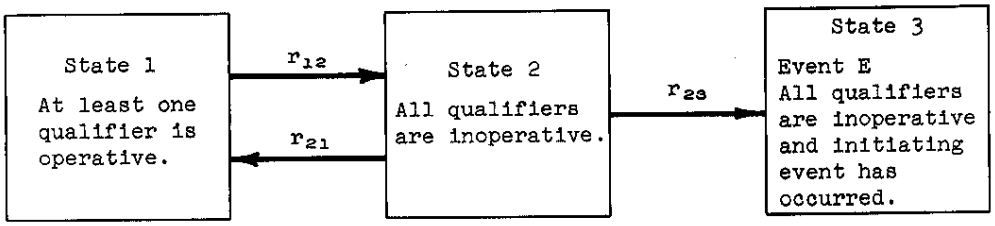
Once the P matrix and the π vector are made up, the solution proceeds by successive multiplication of the vector by the matrix. At the epochs of scheduled inspections and maintenance, the π vector is readjusted to reflect the information gained during the inspection. This readjustment is made as described above. The results printed out at each inspection epoch are: (1) time, (2) probability of complex event during time since last inspection, (3) probability of nonoccurrence of complex event since time zero, (4) $-\ell n(1 - P_\epsilon(T))$, and (5) qualifiers inspected.

Although the situation in which the computer program is most usefully applied are complex, the results can be expressed in the form of equation (6) written here as

$$Pr(E) = 1 - e^{-\alpha t} \tag{28}$$

where $1/\alpha$ is the mean time to occurrence of event E. Thus, one parameter characterizes the complex behavior of a set of events, a set of qualifiers, an inspection and maintenance schedule, and correlations between failures of qualifiers. In other words, α is some function of all of these quantities and characteristics.

The characterization of a safety system by a minimum set of parameters independent of the properties of the events seems desirable. This problem is analyzed in the following manner. Any system, no matter how complex, that has a time behavior of the form of equation (28) is equivalent to a three-state system. By equivalent is meant that the α's for the two processes are the same function of the probability of the initiating event X when this latter probability is the only variable in the processes. Later discussion and examples clarify this meaning of equivalence. The three-state system is shown in the following diagram.



The r_{ij} are the rates or frequencies of the transfer between the three states because the system is now to be considered continuous rather than discrete. States 2 and 3 in the three-state system are identical in all respects to the last two states in the complex system and r_{23} is the frequency of the initiating event identical with that in the complex system. All of the complexities of the first $2^N - 1$ states in the complete system and of the inspection and maintenance schedule have been condensed into a single state and the two parameters r_{12} and r_{21}. Thus, the

objective of finding a minimum set of characterizing parameters has been formally satisfied by showing that any safety system and its associated operating procedures are described by the parameters r_{12} and r_{21}.

The calculation of r_{12} and r_{21} in a specific case where α has been determined by the computer program requires the functional relation between α and r_{12}, r_{21}, and r_{23}. This relation is obtained by starting with the following set of equations.

$$\left.\begin{aligned} \frac{d\pi_1}{dt} &= -r_{12}\pi_1 + r_{21}\pi_2 \\ \frac{d\pi_2}{dt} &= r_{12}\pi_1 - r_{21}\pi_2 - r_{23}\pi_3 \\ \frac{d\pi_3}{dt} &= r_{23}\pi_2 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}\right\} \qquad (29)$$

This set of equations is similar to set (9) except they represent a continuous Markov process rather than a discrete process. These equations will be recognized as the usual equations for first-order reactions. If r_{21} were zero, they would describe a radioactive decay chain $A \rightarrow B \rightarrow C$.

The item of concern in this system is $\pi_3(t)$ which is the probability that the system has arrived at state 3 sometime prior to time t. It is the cumulative distribution function of the random variable, which is the time to occurrence of state 3. As will be seen, $\pi_3(t)$ increases monotonically from a value of zero at $t = 0$ to unity as $t \rightarrow \infty$.

The solution of the set of equations (29) is

$$\pi_3(t) = 1 - \frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha} \qquad (30)$$

where

$$\alpha = \frac{2r_{12}r_{23}}{r_{12} + r_{21} + r_{23} + \sqrt{(r_{12} - r_{23})^2 + r_{21}^2 + 2r_{12}r_{21} + 2r_{21}r_{23}}}$$

and (31)

$$\beta = \frac{r_{12}r_{23}}{\alpha}$$

The details of the solution are omitted here, but substitution of equations (30) and (31) into set (29) verifies that they are indeed solutions.

The fact that equation (30) does not match exactly the form of equation (28) is the result of an initial transient condition introduced into the solutions of equations (29) by assuming that $\pi_1(0) = 1.0$ and $\pi_2(0) = \pi_3(0) = 0$. However, β is always much larger than α in systems of interest so that the second exponential rapidly becomes negligible compared to the first exponential. This relationship is similar to equation (6) and a plot of sample data, as shown in Figure 3, is linear with a slope of α. That is

$$-\ell n[1 - P_\epsilon(t)] = -\ell n \frac{\beta}{\beta-\alpha} + \alpha t \qquad (32)$$

Equation (32) shows that the straight line does not necessarily pass through the origin. Since the primary concern is with the long-term behavior of systems, the effects of the initial conditions are always disregarded and all characterizations are derived from the one parameter, α.

The cumulative distribution function $\pi_3(t)$ has the density function

$$\frac{d\pi_3(t)}{dt} = \frac{\alpha\beta}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right) \qquad (33)$$

The failure rate, which is sometimes used to describe systems, is

$$\frac{d\pi_3(t)/dt}{1 - \pi_3(t)} = \frac{\alpha\beta\left(1 - e^{-(\beta-\alpha)t}\right)}{\beta - \alpha e^{-(\beta-\alpha)t}} \qquad (34)$$

The expected time to the occurrence of state 3 is the item of greatest concern. This quantity is defined as

$$E(T) = \frac{\int_a^1 t\, d\pi_3(t)}{\int_a^1 d\pi_3(t)} \qquad (35)$$

The results of the indicated integration yields

$$E(T) = \frac{\beta^2 - \alpha^2 e^{-(\beta-\alpha)t}}{\alpha\beta\left(\beta - \alpha e^{-(\beta-\alpha)t}\right)} \qquad (36)$$

This expected interval is $(\beta + \alpha)/\alpha\beta$ and $1/\alpha$ for $t = 0$ and $t = \infty$, respectively. The difference between these two values is only a reflection of the initial transient, and the approach to the value $1/\alpha$ is very rapid for systems of interest. In all that follows, $1/\alpha$ will always be used as representative of the process.

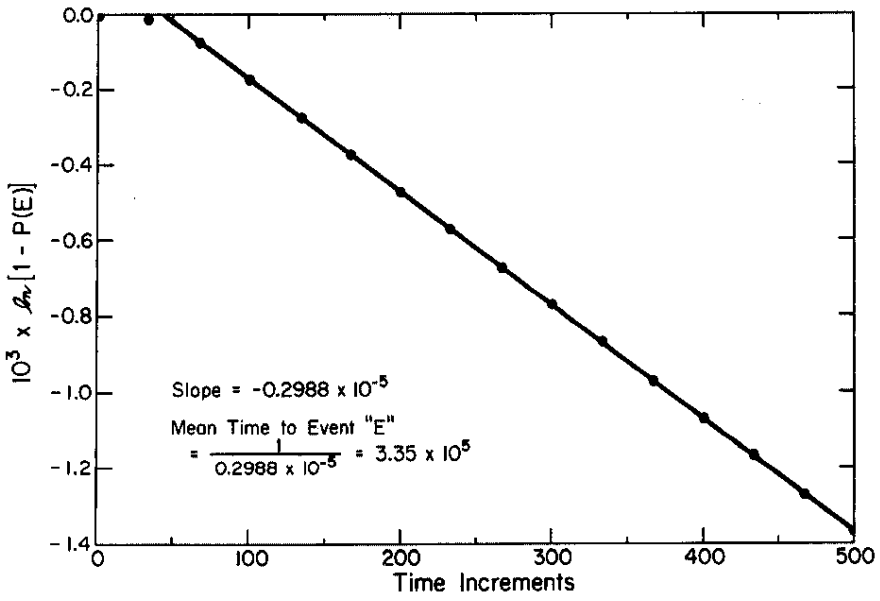


FIG. 3 PROBABILITY OF EVENT "E"

The following numerical examples illustrate how data on models that simulate realistic reactor systems are converted to equivalent three-state systems. The safety system consists of three qualifiers, each of which becomes inoperative, on the average, once every 500 time units. Four variations of the inspection schedule were considered as follows:

Schedule	Description
A	Each qualifier was inspected on a random schedule with an average frequency of once each 100 time units.
B	Each qualifier was inspected simultaneously every 100 time units.
C	Each qualifier was inspected every 100 time units but with a phase shift of 33 time units between each device, i.e., Qualifier 1 - 33, 133, 233, ... Qualifier 2 - 67, 167, 267, ... Qualifier 3 - 100, 200, 300, ...
D	All qualifiers were inspected, on the average, once every 100 time units according to the following schedule: Qualifier 1 - 97, 194, 291, ... Qualifier 2 - 100, 200, 300, ... Qualifier 3 - 103, 206, 309, ...

The following table lists the results of the calculations.

Characteristics of Safety System

Inspection Schedule	Frequency, r_{23} of Initiating Event	Mean Time to Complex Event $1/\alpha$	Constants of Equivalent Three-State System	
			r_{12}	r_{21}
A	2×10^{-3}	$1.16 \times 10^{+3}$		
A	4	0.62	1.28×10^{-4}	2.76×10^{-2}
A	6	0.44		
B	2	3.47		
B	4	1.80	0.73	4.85
B	6	1.25		
C	2	9.70		
C	4	4.93	0.59	11.3
C	6	3.35		
D	2	6.81		
D	4	3.50	0.54	7.13
D	6	2.40		

The r_{12} and r_{21} are derived from the α's calculated by the computer program by use of the relation shown in equation (31). In systems that provide adequate protection, r_{12} is small compared to r_{21} and possibly to r_{23}. This assumption permits r_{12} to be neglected in the denominator of equation (31) and α to be written as approximately

$$\alpha \approx \frac{r_{12} r_{23}}{r_{21} + r_{23}} \tag{37}$$

This equation can be rearranged to

$$\frac{1}{\alpha} = \left(\frac{r_{21}}{r_{12}} \frac{1}{r_{23}} \right) + \frac{1}{r_{12}} \tag{38}$$

Thus, for a safety system in which r_{23} is the only variable, $1/\alpha$ is a linear function of $1/r_{23}$. Further, when $1/\alpha$ is plotted against, $1/r_{23}$,

$$r_{12} = \frac{1}{\text{Intercept}} \tag{39}$$

and

$$r_{21} = \frac{\text{Slope}}{\text{Intercept}} \tag{40}$$

The data for the four systems shown in the preceding table are plotted in Figure 4 to show that the expected linear relation does indeed hold. The constants for the equivalent three-state systems were obtained from these linear relations. Because there are only two parameters, r_{12} and r_{21} to be determined, only two pairs of values for α and r_{23} are required.

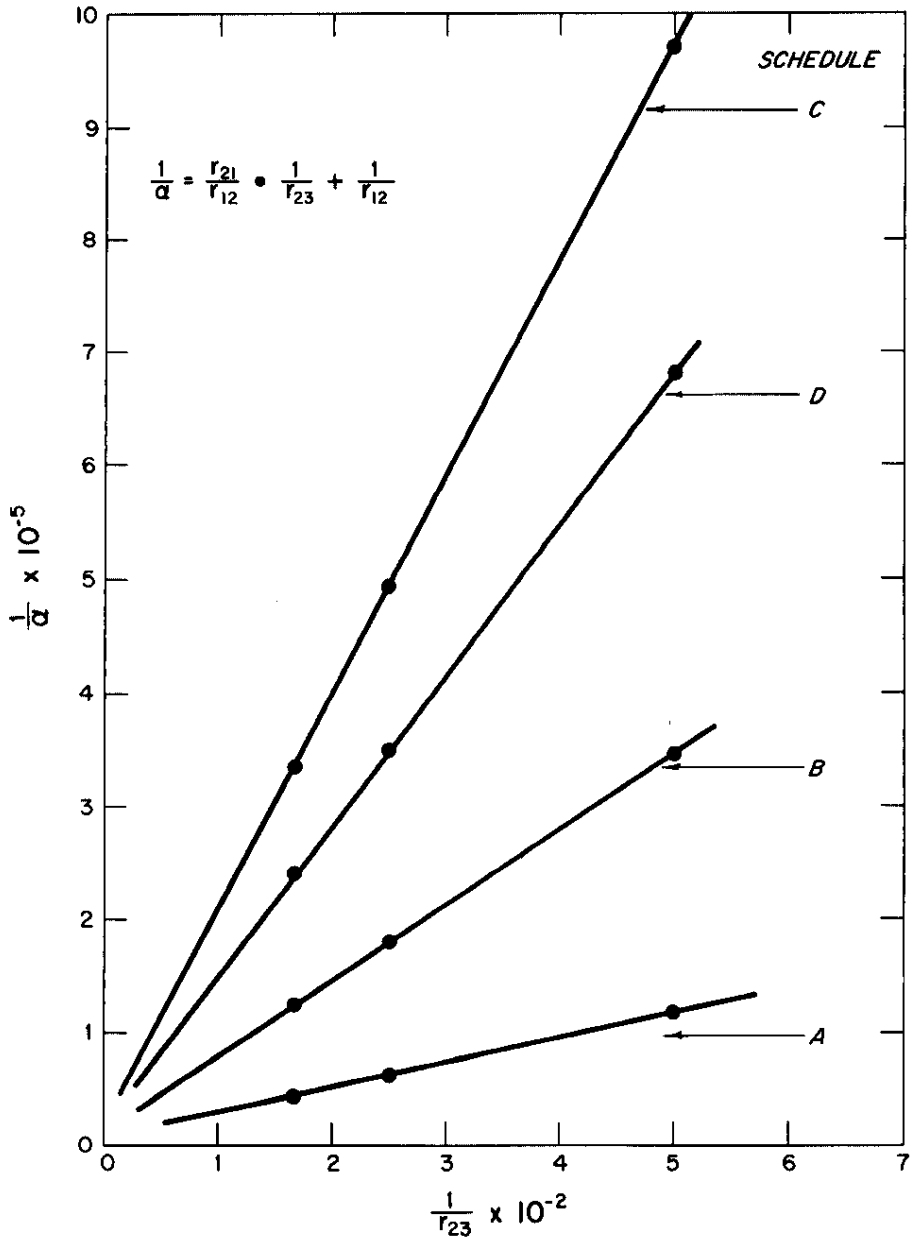


FIG. 4 PLOT OF FUNCTION

FREQUENCIES OF RARE EVENTS

A thorough and satisfactory analysis of the safety of any complex system by these probabilistic methods will require some inputs that are frequencies of rare events. The occurrence of these rare events may lead to severe consequences so that this possibility should not be ignored. Since by definition rare events occur infrequently, the number assigned to the frequency of such an event is based on very little data and is consequently subject to appreciable uncertainty. For example, what frequency is to be assigned to a postulated event that has never been observed to occur?

The following procedures constitute a method for assigning frequencies to rare events on a consistent and statistically valid basis. The intervals between occurrences of events are assumed to have an exponential distribution. The number of events in unit time thus has a Poisson distribution. This assumption is generally regarded as valid for failures of mechanical equipment provided that the equipment is run-in or tested to correct obvious manufacturing defects and is replaced, maintained, or repaired prior to the onset of excessive wear. From an observation of the number, C, of events, in this case, mechanical failures, in a time T, the probability that the true number of events is less than or equal to r is given by,

$$P(\lambda \leqslant r|C) = \int_0^r \frac{\alpha^C e^{-\alpha}}{C!} d\alpha \qquad (41)$$

Equation (41) is derived by the use of Bayes' theorem with the assumption that the process of concern is drawn from a population of Poisson processes that have a uniform distribution of frequencies. This assumption is the equivalent of no preconception of the likely failure rate prior to making the observation. For example, suppose that 10 items of equipment were each operated for five years without a failure, what is it reasonable to assume concerning the true failure rate? In this instance, C equals zero and

$$P(\lambda \leqslant r|0) = \int_0^r e^{-\alpha} d\alpha$$

If the confidence level concerning the assertion with respect to λ is set at 90%, then

$$\int_0^r e^{-\alpha}\, d\alpha = 0.9$$

and

$$r = 2.302$$

Therefore, the probability is 0.90 that the true number of events is equal to or less than 2.302 or the failure rate is equal to or less than 2.303/50 = 0.046 per unit-year. The following table is a short example of confidence limits on the true number of events given various observed values, C.

Upper Confidence Limits for True Number of Events in a Poisson Process

Observed Number of Events, C	Confidence Level			
	0.90	0.95	0.975	0.99
0	2.302	2.996	3.689	4.605
1	3.890	4.744	5.572	6.638
2	5.322	6.296	7.223	8.406
3	6.681	7.754	8.767	10.045
4	7.994	9.154	10.242	11.605

The integral in equation (41) is the χ^2 integral and thus can be found already tabulated in collections of statistical tables, e.g., Pearson and Hartley, Biometrika Tables for Statisticians, Vol. I. The degrees of freedom, ν, in the χ^2 distribution is equal to $\nu = 2(C + 1)$ and χ^2, the value of the parameter, equals $\chi^2 = 2r$. Thus, the value of r can be read from the tables for any tabulated values of P and C.

APPENDIX A

Computer Program INTEG

This version of the computer program was written by Jeffrey L. Piech, an employee in the Applied Mathematics Division at the Savannah River Laboratory during the summer of 1967.

This program calculates, as a function of time, the probability of a complex event that is the occurrence of an initiating event at a time when all safety qualifiers are inoperative. The program can analyze a system consisting of many qualifiers; e.g., 50-100. The effect of inspection and maintenance schedules is accounted for, but all failures of the qualifiers are independent.

Programming Notes

The program works in the following order:

1. Reads the number of problems to be calculated and does the following for each job.

2. Reads number of safety qualifiers, the number of integration steps per unit of time, total time for problem, frequency of initiating event, time interval for printout of results, and an option to allow starting integration at a time other than an inspection epoch.

3. For each safety qualifier, reads failure frequency, time between inspection, time of first inspection.

4. Sets up schedule for inspections of all safety qualifiers and begins numerical integration.

5. At each inspection point the result of the integration is the probability of the complex event during the time interval since previous inspection. If this probability is less than 10^{-14}, the integration is continued until the sum of the probabilities in successive intervals exceeds 10^{-14}. When this condition is met, the sum is subtracted from unity to give the probability of the nonoccurrence of the complex event over the group of intervals. The product of the nonoccurrence is accumulated, and when the probability of occurrence of the complex event is required the product is subtracted from unity.

6. Prints out results which consist of the following:

- Identity of qualifiers inspected during each time interval selected for printout of results
- Time selected for printout of results
- Value of integrand in equation (36)
- Cumulative probability of complex event
- $-\ell n[1 - P(T)]/T$

This latter quantity is the frequency of the equivalent Poisson process.

Input Data Format

Card Type	Column	Format	Description	Comment
A	1-5	I5	Number of problems in set	
B	1-10	I10	Number of qualifiers	
	11-20	I10	Number of integration steps per time unit	Accuracy increases with increase in number of steps
	21-35	D15.0	Total time units of problem	
	36-50	D15.0	Frequency of initiating event	
	51-65	D15.0	Intervals at which results are printed	
	66-70	I5	1, if the initial inspection of any qualifier occurs at time prior to T = 0; otherwise, 0	
C	1-15	D15.0	Failure frequency of qualifier	One card of Type C for each qualifier
	16-30	D15.0	Time between inspections of qualifier	
	31-45	D15.0	Time of first inspection of qualifier	
	46-60	D15.0	If option 1 was used on Card B in column 66-70, then a negative number may be used to indicate an inspection prior to T = 0	

```
      REAL*8 LAMBDA(100),TIME(100),CURNTT(100),PREVTM(100),PROD,CUMPRO,
     1LMBDA,FTIME,DIVPT,T,STPSZE,X,Y,PPRO,SUM,PPROT,TNEXT,TPUT,TDELTA,TR
     1UET,TPUT1,PROD1,TLAST,TMPSUM,SSUM,DLCJMP
      DIMENSION INDEX (100)
      CALL EFTM(40)
C INTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATION
C
C  PROGRAM PERFORMS INTEGRATION OF PROBABILITY DENSITY FUNCTIONS TO DETERMINE
C   THE PROBABILITY OF AN EVENT IN A GIVEN TIME PERIOD. THE DENSITY FUNCTION
C   IS TYPICALLY A NORMALIZED PRODUCT OF DENSITY FUNCTIONS, EACH OF WHICH IS
C   USUALLY DISCONTINUOUS. THE INTEGRATION IS ROUGHLY DONE BY THE
C   TRAPEZOIDAL RULE BETWEEN DISCONTINUITIES OF THE DENSITY FUNCTION. THE
C   PRODUCT OF EACH OF THESE  INTEGRATIONS IS THE DESIRED PROBABILITY.
C
C  INPUT VARIABLES
C
C   NJOB - NUMBER OF JOBS TO BE RUN
C   NFACTR - NUMBER OF PROBABILITY   DENSITY FUNCTIONS TO BE READ IN
C   ISTEP  - NUMBER OF INTERVALS EACH UNIT OF TIME IS TO BE DIVIDED INTO . IN
C                  GENERAL, THE LARGER ISTEP, THE MORE ACCURATE THE RESULT.
C                  HOWEVER, MAKING ISTEP TOO LARGE GREATLY INCREASES RUNNING
C                  TIME AND COULD CAUSE UNDERFLOW.
C  FTIME  - FINAL TIME. FUNCTION IS INTEGRATED FROM ZERO TO FTIME.
C   LMBDA  - PROBABILITY OF ORIGINAL EVENT HAPPENING
C   TPUT    - INTERVALS ON TIME AXIS AT WHICH PROBABILITIES ARE OUTPUTTED
C   IOPT   - OPTION WHICH ALLOWS STARTING INTEGRATION AT SOME OTHER POINT THAN
C                  A DISCONTINUITY.  (SEE PREVTM)
C
C                  FOR EACH JOB THERE ARE NFACTR CARDS, EACH WITH THE FOLLOWING
C
C   LAMBDA(I) -  FAILURE RATE FOR THE I'TH CIRCUIT
C    TIME(I)   -  TIME BETWEEN INSPECTIONS OF I'TH CIRCUIT
C    CURNTT(I) -  TIME OF FIRST INSPECTION OF I'TH CIRCUIT.  (SHOULD BE LESS
C                  THAN TIME(I). IF CURNTT(I) = 0, THEN FIRST INSPECTION
C                  OCCURS AT TIME(I).)
C    PREVTM(I) -  USED IF IOPT = 1. NEGATIVE NUMBER INDICATING WHEN LAST
C                  INSPECTION TOOK PLACE, GIVEN T = 0 AS STARTING TIME.
C
C INTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATIONINTEGRATION
      READ (5,400) NJOB
  400 FORMAT (I5)
      DO 19 NUM = 1, NJOB
      WRITE (6,401) NUM
  401 FORMAT (' JOB NUMBER',I5,/////)
      READ (5,500) NFACTR,ISTEP,FTIME,LMBDA,TPUT,IOPT
  500 FORMAT (2I10,3D15.0,I5)
      WRITE (6,700)
  700 FORMAT (26X,'INPUT DATA')
      WRITE (6,701)
  701 FORMAT (///' NUMBER OF STEPS / UNIT TIME',9X,'FINAL TIME',9X,'LAMB
     1DA')
      WRITE (6,600) ISTEP, FTIME, LMBDA
  600 FORMAT (15XI5,16XD17.10,2XD17.10)
      WRITE (6,702)
```

```
702 FORMAT (///' SAFETY CIRCUIT FAILURE')
    WRITE (6,703)
703 FORMAT (3X,'RATE',14X,'TIME BETWEEN INSPECTIONS',5X,'TIME OF FIRST
   1 INSPECTION')
    WRITE (6,704)
704 FORMAT (/)
    DO 9 I = 1, NFACTR
    READ (5,501) LAMBDA(I), TIME(I), CURNTT(I), PREVTM(I)
501 FORMAT (4D15.0)
    WRITE (6,601) LAMBDA(I), TIME(I), CURNTT(I)
601 FORMAT (1XD10.3,7XD17.3,12XD17.3)
  9 CONTINUE
    WRITE (6,705)
705 FORMAT('1')
    WRITE (6,706)
706 FORMAT (11X,'TIME',13X,'DENSITY FUNCTION VALUE',12X,'CUMULATIVE PR
   1OBABILITY          (-LN(1-P(T)))/T')
    PPRO = 1.DO
    TLAST = 0.
    DIVPT = ISTEP
    STPSZE = 1.DO/DIVPT
    T = -STPSZE/2.DO
    TDELTA = -T
    TPUT1 = TPUT
    DO 10 I = 1, NFACTR
    IF (IOPT-1) 4,5,4
  4 PREVTM(I) = 0.
  5 INDEX(I) = 0
    IF (CURNTT(I) - 10.E-10) 11,11,10
 11 CURNTT(I) = TIME(I)
 10 CONTINUE
    T = STPSZE + T
    SUM = 0.
    SSUM = 0.
 18 TNEXT = CURNTT(1)
    IF(NFACTR.EQ.1) GO TO 41
    DO 20 I = 2, NFACTR
 20 TNEXT = DMIN1(TNEXT,CURNTT(I))
 41 INDEED = 0
    DO 12 I = 1, NFACTR
    IF (CURNTT(I) - TNEXT -.1) 13,13,12
 13 INDEED = INDEED + 1
    INDEX(INDEED) = I
 12 CONTINUE
    INEXT = TNEXT - TLAST + .005
    IEND = INEXT*ISTEP
    DO 14 I = 1, IEND
    PROD = 1.DO
    DO 15 J = 1, NFACTR
    X = LAMBDA(J)*(PREVTM(J) - T)
    Y = DEXP(X)
    PROD = PROD*(1.DO - Y)
```

```
 15 CONTINUE
    PROD = PROD*LMBDA
    X = LMBDA*(T - TNEXT)
    Y = DEXP(X)
    PROD = PROD*Y*STPSZE
    SUM = SUM + PROD
    TRUET = T + TDELTA
    IF (TRUET-TPUT1+.00001) 21,22,22
 22 TPUT1 = TPUT1 + TPUT
    PROD1 = PROD/STPSZE
    TMPSUM = SSUM + SUM*DEXP(LMBDA*(TNEXT-TRUET))
    PPROT = PPRO*(1. - TMPSUM)
    CUMPRO = 1.-PPROT
    IF (CUMPRO - 1.D-14) 32,33,33
 32 CUMPRO = TMPSUM
 33 CONTINUE
    DLCUMP = -DLOG(1.DO - CUMPRO)/TRUET
    WRITE (6,503) TRUET,PROD1,CUMPRO,DLCUMP
503 FORMAT (5XD13.5,12XD20.13,11XD20.13,10XD20.13)
 21 CONTINUE
    T = T + STPSZE
 14 CONTINUE
    WRITE (6,707) (INDEX(I), I = 1, INDEED)
707 FORMAT (' FACTORS INSPECTED ARE  ', 10I5)
    SSUM = SSUM + SUM
    IF (SSUM - 1.D-14) 30,31,31
 31 PPRO = PPRO*(1.DO - SSUM)
    SSUM = 0.
 30 SUM = 0.
    TLAST = TNEXT
    DO 17 I = 1, INDEED
    PREVTM(INDEX(I)) = TNEXT
    CURNTT(INDEX(I)) = TNEXT + TIME(INDEX(I))
 17 CONTINUE
    IF (TNEXT-FTIME) 18,19,19
 19 CONTINUE
    STOP
    END
```

APPENDIX B

Computer Program INSPEC

This version of the computer program was written by James E. Huneycutt, an employee in Applied Mathematics Division at the Savannah River Laboratory during the summer of 1966.

This program calculates, as a function of time, the probability of the complex event which is the occurrence of a potential perturbation at a time when all applicable safety qualifiers are inoperative. The program can analyze a system consisting of as many as seven safety qualifiers and all of the perturbations against which the safety qualifiers provide protection. The effect of inspection and maintenance schedules and possible correlations between failures of actions are accounted for in the computation. During the time intervals between inspections, the operation of the system is simulated by a Markov process. At inspection times, the probabilities of all the possible states are adjusted in accordance with Bayes' theorem.

Programming Notes

Comments are given at the beginning of the program for the use of the more important variables. The program works in the following order:

1. Reads the number of problems to be run, and does the following for each problem.

2. Reads the number of safety qualifiers (NFACT), the total number of time increments to be considered, and the state to be analyzed. (The state to be analyzed is usually left blank; the blank sets the value to $(2^{NFACT} + 1)$ which is the complex event.)

3. Reads the following for each safety qualifier in the system:

   (a) The length of the interval at which the unit is inspected, K. (e.g., every 100 time increments).

   (b) The lag, L, such that: time to first inspection = K-L.

4. Creates the list of all possible states for the system. (The procedure gives $(2^{NFACT} + 1)$ states.)

5. Reads the probability of failure per time increment for each qualifier.

6. Creates the transition (probability) matrix under the assumption that all units are independent and non-self-correcting.

7. Reads the probability per time increment of occurrence of the initiating event.

8. Reads in additions and/or corrections to the transition matrix. (Interdependence of units and self-correction are incorporated here.)

9. Reads whether a printout of the transition matrix is desired; if not, a blank card is sufficient. (Not recommended if NFACT is larger than 3 since the matrix is $(2^{NFACT} + 1)(2^{NFACT} + 1)$.)

10. Prints out the time intervals and lags for inspection and failure probabilities for all safety qualifiers along with the probability of occurrence of the initiating event. The transition matrix is printed out if called for in (9).

11. Reads a card to determine whether the quantity ($P_{12} = \alpha/r_{23}$) is desired. If not, a blank card is sufficient.

12. Sets initial probabilities of all allowed states. These probabilities correspond to all qualifiers operative.

13. Begins computation and does the following for each time increment:

    (a) Determines probabilities of all allowed states at the beginning of the increment.

    (b) Determines which qualifier (if any) are inspected at this increment.

    (c) Prints out results when any qualifiers are inspected.

    (d) Readjusts values of the probabilities of all allowed states if there is an inspection.

    The results consist of:

    - The number of the time increment
    - Probability of complex event during interval since last inspection
    - Probability of nonoccurrence of complex event since t = 0
    - Qualifiers inspected

Input Data Format

Card Type	Column	Format	Description	Comment
A	1-5	I5	Number of problems to be run	
B	1-5	I5	Number of qualifiers to be inspected	≤ 7
	6-10	I5	Number of time increments in problem	
	11-15	I5	State to be analyzed	Usually blank
C	1-5	I5	Interval between inspections of qualifier	One for each qualifier
	6-10	I5	Lag	
D	1-10	D10.5	Probability per time increment of failure of qualifier 1	
	11-20	D10.5	Probability per time increment of failure of qualifier 2	
	.	.		
	.	.		
	.	.		
	.	.		
E	1-10	D10.5	Probability per time increment of occurrence initiating event	
F	1-80	A4	"NO ADDITIONS"	Pick one Don't use "ADDITIONS 0"
	or			
	1-10	A4	"ADDITIONS"	
	16-20	I5	Number of additions	
G	1-5	I5	Index I	Include this only if "ADDITIONS" card is used above.
	6-10	I5	Index J	
	11-20	D10.5	Probability of going from State I to State J	
H	1-80	A4	"PRINT OUT MATRIX" or "DO NOT PRINT OUT MATRIX"	Pick one
I	1-80	A4	"PRINT OUT P12" or "DO NOT PRINT OUT P12"	Pick one

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C                        I N S P E C
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C         P(I,J) IS PROBABILITY OF GOING FROM STATE I TO STATE J
C         A(I) IS PROBABILITY OF BEING IN STATE I
C         KK(I,J) DEFINES THE STATES (I),  KK(I,J) IS 0 IF UNIT J IS
C              OPERATIVE AND 1 IF UNIT J IS INOPERATIVE
C         B(I) IS WORK MATRIX ASSOCIATED WITH A
C         PP(I) IS PROBABILITY OF UNIT I BEING INOPERATIVE
C         K(I) IS INTERVAL AT WHICH UNIT I IS INSPECTED
C         L(I) IS LAG TIME FOR UNIT I
C
C                          FOR EACH TIME INTERVAL
C         D(I) IS 0 IF UNIT I IS INSPECTED AND 1,IF NOT
C         E(I) IS A WORK MATRIX DETERMINING HOW STATE I IS EFFECTED
C         C IS THE DENOMINATOR IN BAYES' THEOREM
       REAL*8 P(150,150),A(150),B(150),PP(10),AB(150),C,CC,BA,VAL,DLCUP
       COMMON P,A,B,PP,AB,C,CC,BA,VAL,D(10),DD(10),E(150),KK(150,10),L(10
      1),K(10),ML(10)
       DATA  TEST/'ADDI'/,PRINT/'PRIN'/
  101 FORMAT (3I5)
  102 FORMAT (2I5)
  103 FORMAT (8D10.5)
  104 FORMAT ( A4,11X,I5)
  105 FORMAT (2I5,D10.5)
  106 FORMAT (A4)
  107 FORMAT (10F10.6)
  201 FORMAT (1H1,10X,22HSCEDULE FOR INSPECTION//15X, 4HUNIT,16X,
     1  8HINTERVAL,12X,3HLAG,20X,'PROBABILITY OF FAILURE'/)
  202 FORMAT (12X,I5,18X,I5,11X,I5,25X,F10.6)
  203 FORMAT (76X,F10.6//////)
  204 FORMAT (4X,I10,2(6X,D14.8),8X,D14.8,10X,10I2)
  205 FORMAT (1H1,30X, 'ANALYSIS OF STATE',10I3)
  206 FORMAT (////19X,'  PROBABILITY OF',7X,'PROBABILITY OF')
  207 FORMAT (6X, 'TIME PERIOD         OCCURRENCE        NON-OCCURRENCE',
     1         7X,'-LN(1-P(T))      TYPE OF INSPECTION'/)
  208 FORMAT (1H+,114X,F15.13)
       READ (5,102) MMMM
       DO 999 LLLL = 1,MMMM
       CC = 1
       READ (5,101) NFACT,NNNN,JJ
       DO 5 I = 1,NFACT
     5 READ (5,102) K(I),L(I)
       WRITE (6,201)
       N = 2**NFACT
       NP1 = N+1
       IF (JJ .EQ. 0 ) JJ = NP1
       DO 404 I = 1,NP1
       DO 404 J = 1,NP1
   404 P(I,J) = 0
C         CREATE STATES
       DO 1 I = 1,N
       M = I - 1
       DO 1 J = 1,NFACT
       KK(I,J) = MOD(M,2)
     1 M = M/2
       DO 4 J = 1,NFACT
     4 KK(NP1,J) = 0
```

```
C         CREATE PROBABILITY MATRIX
      AB(NP1) = 0
      READ (5,103) (PP(J),J = 1,NFACT)
      DO 300 I = 1,N
      IP1 = I + 1
      AB(I) = 0
      DO 301 J = IP1,N
      AM = 0
      BA = 0
      DO 302  LL = 1,NFACT
      MA = KK(J,LL) - KK(I,LL)
      IF (MA.EQ.1)BA = PP(LL)
  302 AM = AM +IABS(MA)
      IF (AM.NE.1.) BA = 0
      AB(I) = AB(I) + BA
  301 P(I,J) =BA
  300 CONTINUE
      NM1 = N-1
      DO 304 J = 1,NM1
  304 P(J,NP1) = 0
      READ (5,103) P(N,NP1)
      DO 7 I = 1,NFACT
    7 WRITE (6,202) I,K(I),L(I),PP(I)
      WRITE (6,203) P(N,NP1)
      AB(N) = P(N,NP1)
      DO 305 J = 1,N
  305 P(NP1,J) = 0
      READ (5,104) TSET,NOM
      IF ( TSET .NE. TEST ) GO TO 401
      DO 402 I = 1,NOM
      READ (5,105)M,J,VAL
      AB(M) = AB(M) + VAL - P(M,J)
  402 P(M,J) = VAL
  401 CONTINUE
      DO 306 I = 1,NP1
  306 P(I,I) = 1. - AB(I)
      READ (5,106) TSET
      IF ( TSET .NE. PRINT ) GO TO 403
      DO 303 I = 1,NP1
  303 WRITE (6,107)  (P(I,J),J=1,NP1)
  403 CONTINUE
      READ (5,106) TSET
      A(1) = 1
      DO 3 I = 2,NP1
    3 A(I) = 0
      WRITE (6,205) (KK(JJ,J),J = 1,NFACT)
      WRITE (6,206)
      WRITE (6,207)
C        BEGIN TIME INTERVALS
      DO 999 KKK = 1,NNNN
C        DETERMINE NEW PROBABILITIES
      DO 6 I = 1,NP1
      B(I) = 0
      DO 6 J = 1,NP1
    6 B(I) = B(I) + A(J)*P(J,I)
C        DETERMINE WHICH UNITS ARE INSPECTED
      DO 29 I = 1,NFACT
      D(I) = 1
      MP = KKK + L(I)
```

```
   29 IF (MOD(MP,K(I)).EQ.0) D(I) = 0
      II = 0
      DO 28 J=1,NFACT
   28 II = II + D(J)
      IF (II .EQ. NFACT ) GO TO 998
      I = II
      CC = CC * (1. - B(JJ) )
      DLCUP=-DLOG(CC)
      DO 20 J = 1,NFACT
   20 ML(J) = D(J)
C        PRINT OUT RESULTS
      WRITE (6,204) KKK,B(JJ),CC,DLCUP,(ML(J),J=1,NFACT)
      IF ( TSET .NE. PRINT ) GO TO 333
      BA = DLOG(CC)/(KKK*P(N,NP1) )
      WRITE (6,208) BA
  333 CONTINUE
C        DETERMINE HOW EACH STATE IS EFFECTED
      DO 16 J= 1,N
      E(J) = 0
      DO 16 II = 1,NFACT
   16 E(J) = E(J) + (1.-D(II))*(D(II) - KK(J,II))**2
      DO 19 J = 1,N
      IF (E(J).NE.0.)DD(J) = 0
      IF (E(J).EQ.0.)DD(J) = 1
   19 CONTINUE
      DD(NP1) = 0
C        CREATE DENOMINATOR FOR BAYES' THEOREM
      C = 0
      DO 17 J= 1,NP1
   17 C = C + DD(J)*B(J)
      IF (C.EQ.0.) GO TO 998
      DO 18 J = 1,NP1
C        CREATE PROBABILITIES DETERMINED BY INSPECTION
   18 A(J) = DD(J)*B(J)/C
      GO TO 999
  998 DO 30 M=1,NP1
   30 A(M) = B(M)
  999 CONTINUE
      STOP
      END
```