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GENERALIZED LINEAR REGRESSION ANALYSIS

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GENERALIZED LINEAR REGRESSION ANALYSIS

by

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ABSTRACT

A generalized computer program was developed to fit selected functions to experimental data by the method of least squares. This program is characterized by a wide selection of functional forms for the independent and dependent variables, an extensive listing of descriptive statistics, and a simple input form that facilitates program use.

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INTRODUCTION

In general, there are two reasons for estimating the parameters of an analytical function from a set of observed values. In one case, the fitted function is to be used for interpolation and/or extrapolation. For this use, the functional form is of interest only to the extent that it is both adequate and simple. In the second case, the parameters of the fitted function have physical meaning and the fitting determines these physical values. For this use, the functional form is determined by the physical situation and may be simple or extremely complex.

Whenever the functional form is linear in the parameters (the regression coefficients), a linear least squares program can be used. The computer program described in this report has the special advantages of:

- Wide selection of functional forms for the independent and dependent variables
- Extensive listing of descriptive statistics
- Simple procedure for the user

Appendix A contains the mathematical formulation of the program. Appendix B contains the detailed input instructions and the routine input form. Appendix C contains a sample output for the problem in Appendix B.

SUMMARY

This computer program obtains a least squares fit to K sets of observed values $(Y, X_1, \ldots, X_6)$ by fitting the following linear form:

$$\hat{H}(Y) = \sum_{i=1}^{n} b_i F_i$$

where $\hat{H}(Y)$ is a selected function of the dependent variable Y, the b_i are the regression coefficients to be determined, and the F_i are selected functions of the independent variables X_j. This program has upper limits of 6 independent variables, 20 terms, and 500 sets of observed data. These limits could be increased to accommodate larger problems. The following statistics are calculated: the means, the standard deviations, the pairwise correlations, and the maximum and minimum values of the dependent and independent variables; the regression coefficients b_i and their standard errors; the multiple correlation coefficient, the standard error of estimate, and the ratios of the calculated value to the observed value of the selected function of the dependent variable. A plot of the observed value and calculated value as a function of the first independent variable is also provided under user control.

DISCUSSION

MATHEMATICAL FORMULATION

This program calculates from K sets of observed values $(Y_k, X_{k1}, \ldots, X_{kj})$ that set of coefficients $(b_1, \ldots, b_n)$ of the linear form in b_i:

$$\hat{H}_k = b_1F_{k1} + b_2F_{k2} + \ldots + b_nF_{kn}$$

which minimizes

$$Z = \sum_{k=1}^{K} W_k(H_k - \hat{H}_k)^2$$

where

H_k = value of a specified function for the k^{th} observed value of the dependent variable Y_k

F_{ki} = value of a specified function for the k^{th} observed values of the independent variables X_{kj}

W_k = a normalization or weighting factor that equalizes the variances of all H_k

This minimum value of Z and the corresponding b_i are obtained by the usual methods of the differential calculus by solving the following set of n equations in n unknowns (see Appendix A for details):

$$\frac{\partial Z}{\partial b_i} = 0; \qquad i = 1, \ldots, n \qquad (1)$$

If this system is nonsingular, then a unique set of coefficients b_i exists.

STATISTICAL ASPECTS OF THE PROBLEM

For each given value of F_{ki}, the observed H_k are assumed to be normally distributed with a mean of $\sum_{i=1}^{n} b_i F_{ki}$ and a variance $S_k{}^2$. If the variances are not equal for all values of F_{ki}, then the weighting factor W_k is used such that $S_k{}^2/W_k = S^2$ is constant.

This situation is illustrated in Figure 1 for the case

$$\hat{H}_k = b_1 F_{k1} + b_2 F_{k2}$$

where

$$F_{k1} = 1$$

$$F_{k2} = X_{k1}$$

or

$$H_k = b_1 + b_2 X_{k1}$$

At each of the three values X_{11}, X_{21}, and X_{31}, the mean of the observed H_k is $\hat{H}_k = b_1 + b_2 X_{k1}$. This is the mean about which the observed values of H_k are normally distributed with a variance S^2. The small bell-shaped curves at each value of $\hat{H}_k$ represent normal distributions with equal variances.

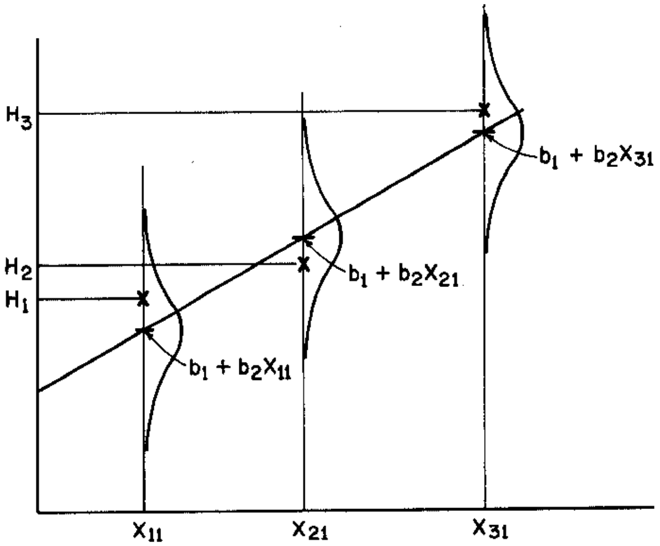


FIG. 1 ILLUSTRATION OF STATISTICAL ASPECTS OF LEAST SQUARES

INPUT FOR THE COMPUTER PROGRAM

The input for this program is listed on a routine form (Appendix B) that allows up to 6 independent variables, 20 terms in the fitted equation, and 500 sets of observed data. Each term involving independent variables is formed from multiplicative combinations of the following functional forms:

Independent Variables	Dependent Variables
1	$[Y]^E$
$[X_j]^E$	$[EXP(Y)]^E$
$[EXP(X_j)]^E$	$[LN(Y)]^E$
$[LN(X_j)]^E$	$[SIN(Y)]^E$
$[SIN(X_j)]^E$	$[COS(Y)]^E$
$[COS(X_j)]^E$	
$[X_j]^E LN(X_j)$	
$[X_j]^E EXP(X_j)$	

In performing the minimization of Z, the user has the option of specifying the following weighting functions:

Option	Result
0	$W_k = 1$
1	W_k read in
2	$W_k = 1/H(Y_k)$
3	$W_k = 1/[H(Y_k)]^2$

OUTPUT FOR THE COMPUTER PROGRAM

The standard output includes the following:

- Problem title
- Model definition (functional form of the fitted equation)
- Regression coefficients, their standard errors, and their "t" values
- Constant term, if present in the model
- Standard deviation and mean of each term in the model
- Simple correlations between the terms F_i and H
- Multiple correlation coefficient, whose square gives the percentage of original variation in the data, which is accounted for by the fit of the model
- Standard error of estimate

The following output may be obtained at the option of the user:

- Analysis of variance for the fit
- Table of simple pairwise correlations, means, standard deviations, maximum values, minimum values, and ranges of the original dependent and independent variables before functional transformation
- Listing of the value of H_k, the calculated value of $\hat{H}_k$, the ratio of the former to the latter, the original independent variables X_{kj}, and the W_k of each data point
- Plot of the observed H_k, and the calculated $\hat{H}_k$ versus the first independent variable

SUBSETS OF THE DEFINED MODEL

Once a given model has been selected, any subset of the model's terms may be selected for subsequent fits, without reading in a new set of data. Each subset of terms is defined on the routine form (Appendix B). The only restrictions are that the dependent variable is fixed and if a constant term is included in the original model, all subsets must also contain a constant. This problem selection feature is described more fully on the input form (Appendix B).

APPENDIX A

DETAILED MATHEMATICAL FORMULATION

Definitions

$$\sum_k (W_k F_{ki} F_{km})' = \sum_k W_k F_{ki} F_{km} - \frac{\sum_k W_k F_{ki} \sum_k W_k F_{km}}{\sum_k W_k}$$

$$S_{HH} = \sum_k (W_k H_k H_k)'$$

$$S_{im} = \sum_k (W_k F_{ki} F_{km})'$$

$$\overline{H} = \frac{\sum_k W_k H_k}{\sum_k W_k}$$

$$\overline{F}_i = \frac{\sum_k W_k F_{ki}}{\sum_k W_k}$$

$$r_{im} = \frac{\sum_k (W_k F_{ki} F_{km})'}{\sqrt{S_{ii} S_{mm}}}$$

C_{im} = inverse element corresponding to r_{im}

$$B_i = b_i \sqrt{\frac{S_{ii}}{S_{HH}}}$$

Mathematical Development

Given the problem of minimizing

$$Z = \sum_{k=1}^{K} W_k (H_k - \hat{H}_k)^2$$

where

$$\hat{H}_k = \sum_{i=1}^{n} b_i F_{ki}$$

then

$$\frac{\partial Z}{\partial b_i} = \sum_k W_k H_k \frac{\partial \hat{H}_k}{\partial b_i} - \sum_k W_k \hat{H}_k \frac{\partial \hat{H}_k}{\partial b_i} = 0; \qquad i = 1, \ldots, n$$

or a set of n equations in n unknowns

$$\sum_k W_k H_k F_{ki} = \sum_k W_k b_1 F_{k1} F_{ki} + \ldots + \sum_k W_k b_n F_{kn} F_{ki}; \qquad i = 1, \ldots, n \tag{A-1}$$

Case I – Constant term.

If $F_{ki} = 1$ for some $i = r$; $k = 1, \ldots, K$; then

$$b_r = \overline{H} - \sum_{\substack{i=1 \\ i \neq r}}^{n} b_i \overline{F}_i \tag{A-2}$$

Substituting (A-2) into the other (n-1) equations (A-1)

$$\begin{aligned}\sum_k (W_k H_k F_{ki})' &= \sum_k (W_k F_{k1} F_{ki})' b_1 + \ldots \\ &+ \sum_k (W_k F_{k,r-1} F_{ki})' b_{r-1} + \sum_k (W_k F_{k,r+1} F_{ki})' b_{r+1} \\ &+ \ldots + \sum_k (W_k F_{kn} F_{ki})' b_n; \quad i = 1, \ldots, r-1, r+1, \ldots n\end{aligned} \tag{A-3}$$

Scaling the matrix of coefficients $\sum_k (W_k F_{km} F_{ki})'$ by dividing by $\sqrt{S_{ii} S_{mm}}$ and the right-hand side vector $\sum_k (W_k H_k F_{ki})'$ by $\sqrt{S_{ii} S_{HH}}$ gives

$$\begin{bmatrix} 1 & r_{12} \cdots & r_{1,n-1} & r_{1n} \\ r_{21} & 1 & \cdot & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ r_{n1} & r_{n2} \cdots & r_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_n \end{bmatrix} = \begin{bmatrix} r_{1H} \\ r_{2H} \\ \cdot \\ \cdot \\ \cdot \\ r_{nH} \end{bmatrix} \tag{A-4}$$

Although the notation in the matrix and vectors of (A-4) implies that the dimension is n, the quantities subscripted r are missing in accordance with the preceding equation (A-3) so that the actual dimension is n-1.

After inverting and solving for B_i, the regression coefficients are

$$b_i = B_i\sqrt{\frac{S_{HH}}{S_{ii}}} \quad , \; i\neq r \qquad \text{(A-5)}$$

and the constant term b_r is given by (A-2).

S_{HH} = total sum of squares with degrees of freedom equal K-1

$RES = S_{HH} - \sum_{\substack{i=1\\ i\neq r}}^{n-1} b_i S_{ii}$ = residual sum of squares with degrees of freedom equal K-n

DF_{RES} = degrees of freedom of residual sum of squares

$\hat{S} = \sqrt{\frac{RES}{DF_{RES}}}$ = standard error of estimate

$\underset{i\neq r}{S(b_i)} = \hat{S}\sqrt{\frac{C_{ii}}{S_{ii}}}$ = standard error of regression coefficients

$S(b_r) = \hat{S}\sqrt{\frac{1}{\sum_k W_k} + \sum_{im}\sum \bar{F}_i\bar{F}_m \frac{C_{im}}{S_{im}}}$ = standard error of constant

<u>Case II</u> — No constant term.

The procedure for Case II is the same as Case I except that $\sum_k (W_k H_k F_{ki})'$ is replaced by $\sum_k W_k H_k F_{ki}$ and $\sum_k (W_k F_{km} F_{ki})'$ by $\sum_k W_k F_{km} F_{ki}$.

In this case, there is no b_r by definition.

APPENDIX B

DETAILED INPUT INSTRUCTIONS

The routine input form (Figure 2) used with this program is composed of five major sections: (1) job control cards, (2) identification card, (3) term definition cards, (4) subset definition cards, and (5) data cards. A detailed description of their use follows:

Job Control Cards

Four OS/360 job control cards (excluding/* at the end) are necessary. Of the four, the user need fill out only the two standard JOB cards with the desired entries.

Identification Card

(one card per set of data - all entries right justified)

Card Column	Identification	Remarks
1-48	Problem title	Up to 48 alphanumeric characters in length
49-51	Number of data points	Maximum - 500
53	Number of independent variables	Maximum - 6
55-56	Number of terms in fit including constant	Maximum - 20
58	Weight option	0 - $W_k = 1$ 1 - W_k read in 2 - $W_k = 1/H(Y_k)$ 3 - $W_k = 1/[H(Y_k)]^2$
60	Number of subsets of the defined model to be fitted	n = 0,1 - defined model only n = 2-9 - defined model and n-1 other subsets will be fit Maximum - 9
62	Simple correlation between input variables	0 - simple correlations not calculated 1 - simple correlations calculated

Identification Card (continued)

Card Column	Identification	Remarks
64	Table of input data, calculated function, and ratio for each data point	0 - table is printed 1 - table is not printed
66	Analysis of variance table	0 - table is not printed 1 - table is printed
68	Plotting option	0 - not plotted 1 - plotted

Term Definition Cards

A term definition card is required for each set of data to describe the functional transformation desired for the dependent variable.

In addition, a card is required to define each term of the model because each term is made up of available functional transformations of the independent variables. The desired transformations are selected by specifying the one-character conversion code from the following:

Conversion Code	Function
0	1 (constant term)
1	$[X_j]^E$, $[Y]^E$
2	$[EXP(X_j)]^E$, $[EXP(Y)]^E$
3	$[LN(X_j)]^E$, $[LN(Y)]^E$
4	$[SIN(X_j)]^E$, $[SIN(Y)]^E$
5	$[COS(X_j)]^E$, $[COS(Y)]^E$
6	$[X_j]^E LN(X_j)$
7	$[X_j]^E EXP(X_j)$

The exponent E is the power to which the transformed variable is to be raised and may have as many as five characters with decimal point, i.e., -5.12, 3.32. All unused independent variables and unused terms are left blank. A constant term can be included in the model by selecting conversion code zero (blank) for all independent variables.

OS/360 LINEAR LEAST SQUARES PROGRAM

This routine obtains a least squares fit to K experimental data points $(Y, X_1, X_2 \cdots X_6)$ by fitting the following linear form:

$$H(Y) = \sum_{I}^{NTRMS} b(I)\ F(I)$$

Where H(Y) is a specified function of the dependent variable Y raised to a specified power, b(I) is a parameter to be determined, and F(I) is the product of specified functions of the independent variables X(J) raised to specified powers.

JOB CONTROL CARDS

```
//FDK33208 JOB (3208,L029,006,04,0000,,,8277-
//S,L5365-01,T,03),                                C
//          'F ' D  KNIGHT             ',MSGLEVEL=1
//LSTSQR   EXEC  L5464
//GO.SYSIN DD  *
```

IDENTIFICATION CARD (ONE REQUIRED PER K DATA POINTS)

```
1   SULFURIC ACID CONVERSI      Problem
25  ON                          Title
```

49	30	Number of experimental values (limit 500)
52	2	Number of independent variables (limit 6)
54	4	Number of terms in fit including constant (limit 20)
57	0	Weight option: 0-Wj=1; [1-Wj read in] ; 2-Wj=1/H(Yj); 3-Wj=1/[H(Yj)]^2
59	2	Number of subsets of the defined model to be fit - usually 1
61	1	Simple correlation between variables: 0-not calculated; 1-calculated
63	0	Table of ratios: 0-printed; 1-not printed
65	1	Analysis of variance table: 0-not printed; 1-printed
67	1	Plotting option: 0-no plot; 1-plot

Right justify all entries in columns 49-68

EXPONENTS AND CONVERSION CODES FOR TERM DEFINITION

Conversion code C:

Term 0 (dependent variable)

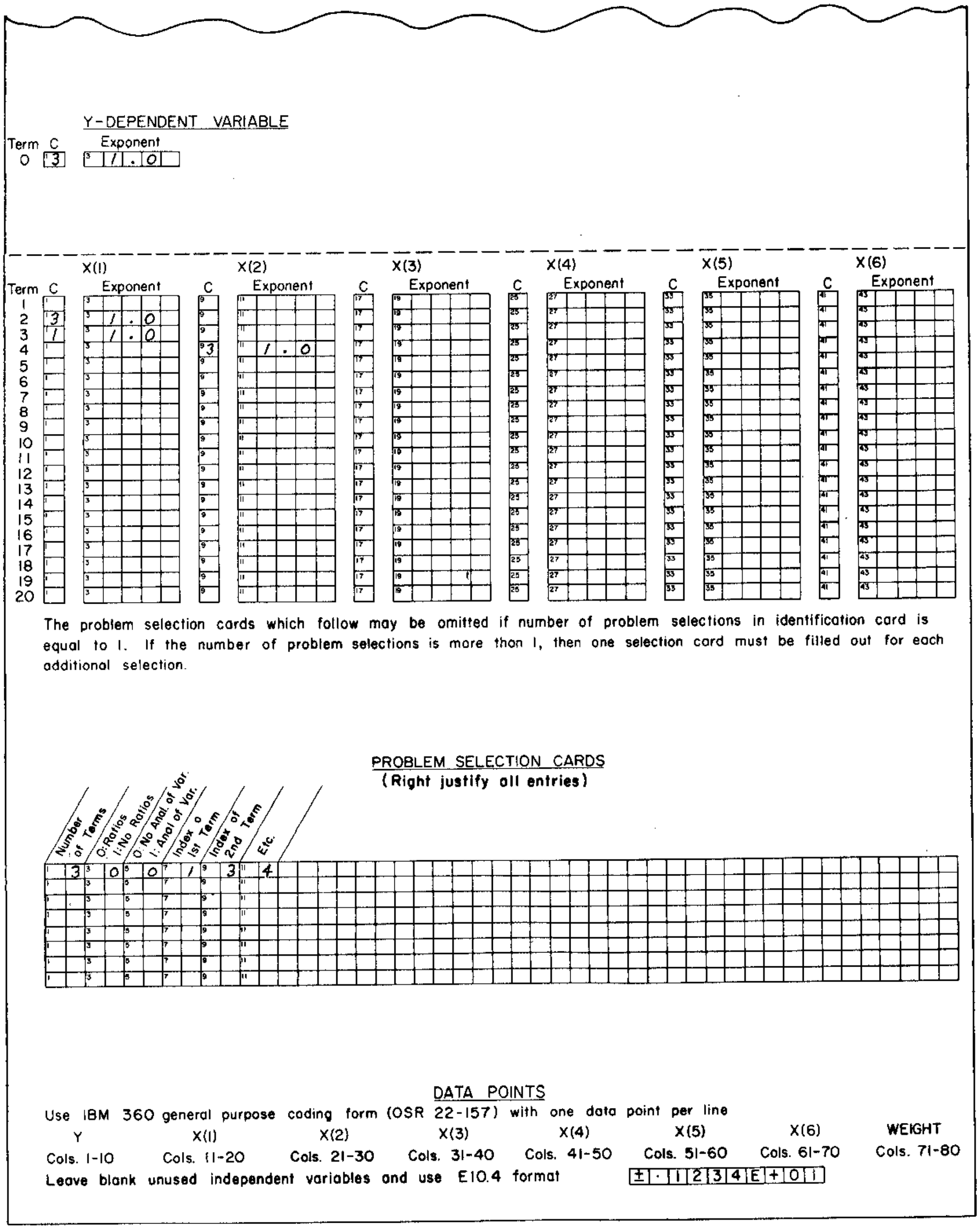
C	Function	C	Function	C	Function
1	Y^E	2	$[EXP(Y)]^E$	3	$[LN(Y)]^E$
4	$[SIN(Y)]^E$	5	$[COS(Y)]^E$		

Term 1-20 (independent variable)

C	Function	C	Function	C	Function
0	1	1	$[X(J)]^E$	2	$[EXP(X(J))]^E$
3	$[LN(X(J))]^E$	4	$[SIN(X(J))]^E$	5	$[COS(X(J))]^E$
6	$[X(J)]^E\ LN(X(J))$	7	$[X(J)]^E\ EXP(X(J))$		

Include decimal points with exponents and leave blank unused independent variables and unused terms. A constant can be input by leaving the line for the indicated term blank.

FIG. 2 ROUTINE INPUT FORM



Y-DEPENDENT VARIABLE

Term	C	Exponent
0	3	1.0

Term	C	X(1) Exponent	C	X(2) Exponent	C	X(3) Exponent	C	X(4) Exponent	C	X(5) Exponent	C	X(6) Exponent
1												
2	3	1.0										
3	1	1.0										
4			3	1.0								
5												
6												
7												
8												
9												
10												
11												
12												
13												
14												
15												
16												
17												
18												
19												
20												

The problem selection cards which follow may be omitted if number of problem selections in identification card is equal to 1. If the number of problem selections is more than 1, then one selection card must be filled out for each additional selection.

PROBLEM SELECTION CARDS
(Right justify all entries)

Number of Terms	0:Ratios 1:No Ratios	0:No Anal. of Var. 1:Anal. of Var.	Index of 1st Term	Index of 2nd Term	Etc.
3	0	0	1	3	4

DATA POINTS

Use IBM 360 general purpose coding form (OSR 22-157) with one data point per line

Y	X(1)	X(2)	X(3)	X(4)	X(5)	X(6)	WEIGHT
Cols. 1-10	Cols. 11-20	Cols. 21-30	Cols. 31-40	Cols. 41-50	Cols. 51-60	Cols. 61-70	Cols. 71-80

Leave blank unused independent variables and use E10.4 format ± . 1 2 3 4 E + 0 1

FIG. 2 Continued

Problem Selection Cards

These cards define the subsets of the defined model which are to be fitted. If the only fit desired is the defined model, then these cards are omitted. One problem selection card is necessary for each subset fit required. A maximum of nine fits are possible.

Card Column	Identification	Remarks
1-2	Number of terms in subset to be fitted	Maximum - 19
3-4	Table of input data, calculated function, and ratio for each data point	0 - table is printed 1 - table is not printed
5-6	Analysis of variance table	0 - table is not printed 1 - table is printed
7-8	Index of first term in subset relative to defined model	
9-10	Same for second term	
11-12	etc.	

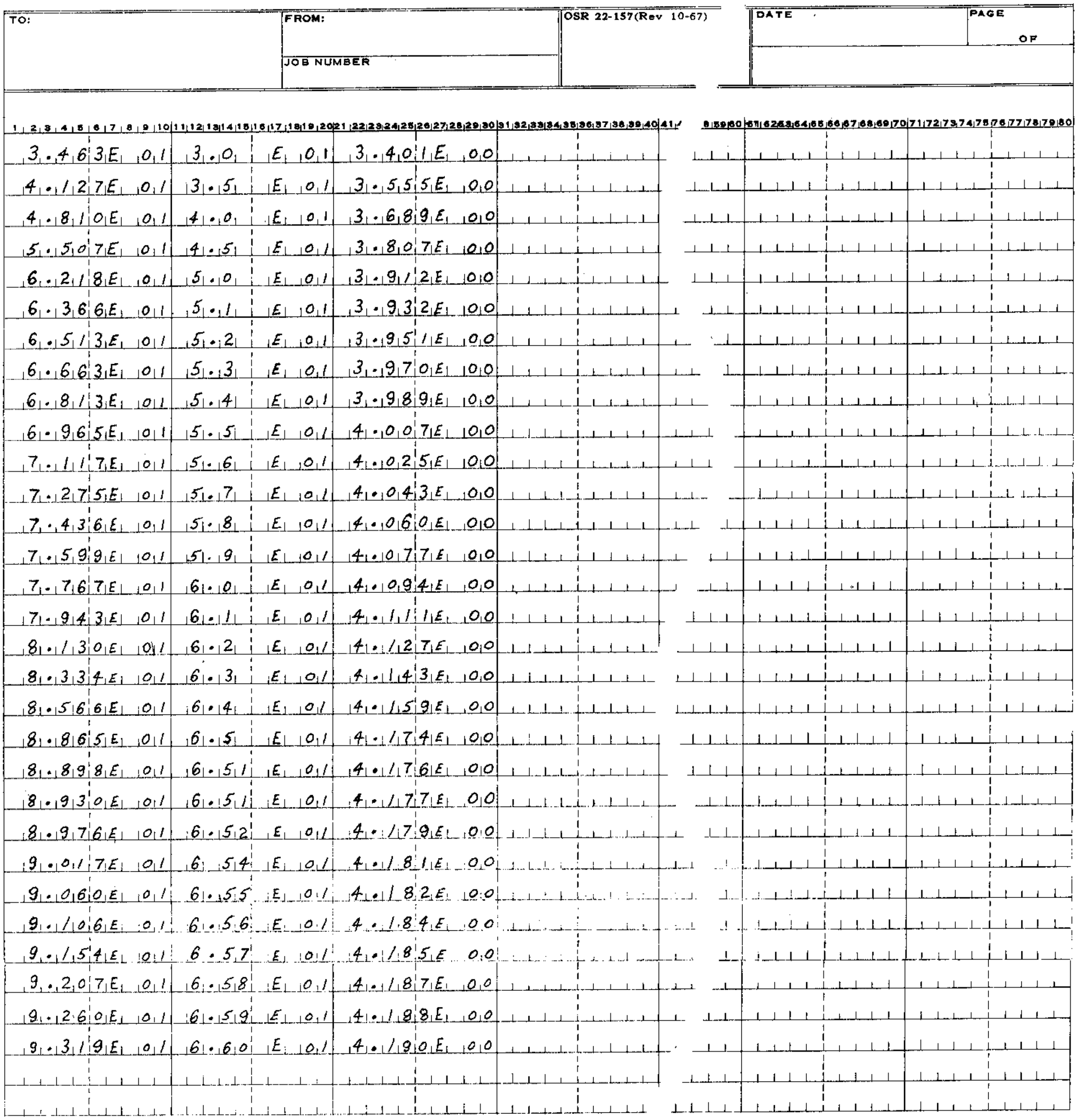
Data Points

The observed values of the dependent and independent variables are listed one set per card in the following order: Y, X(1), X(2), ..., X(6) and WEIGHT by an 8E10.4 format. If WEIGHT is left blank, the WEIGHT is chosen from one of the previously defined weighting options.

In Figure 2, the routine input form is filled out for the following problem:

$$LN(Y) = b_1 + b_2LN(X_1) + b_3X_1 + b_4LN(X_2).$$

The data for the problem are given in Figure 3.



TO:

FROM:

JOB NUMBER

OSR 22-157(Rev 10-67)

DATE

PAGE OF

3.463E 01	3.0 E 01	3.401E 00
4.127E 01	3.5 E 01	3.555E 00
4.810E 01	4.0 E 01	3.689E 00
5.507E 01	4.5 E 01	3.807E 00
6.218E 01	5.0 E 01	3.912E 00
6.366E 01	5.1 E 01	3.932E 00
6.513E 01	5.2 E 01	3.951E 00
6.663E 01	5.3 E 01	3.970E 00
6.813E 01	5.4 E 01	3.989E 00
6.965E 01	5.5 E 01	4.007E 00
7.117E 01	5.6 E 01	4.025E 00
7.275E 01	5.7 E 01	4.043E 00
7.436E 01	5.8 E 01	4.060E 00
7.599E 01	5.9 E 01	4.077E 00
7.767E 01	6.0 E 01	4.094E 00
7.943E 01	6.1 E 01	4.111E 00
8.130E 01	6.2 E 01	4.127E 00
8.334E 01	6.3 E 01	4.143E 00
8.566E 01	6.4 E 01	4.159E 00
8.865E 01	6.5 E 01	4.174E 00
8.898E 01	6.51 E 01	4.176E 00
8.930E 01	6.51 E 01	4.177E 00
8.976E 01	6.52 E 01	4.179E 00
9.017E 01	6 54 E 01	4.181E 00
9.060E 01	6.55 E 01	4.182E 00
9.106E 01	6.56 E 01	4.184E 00
9.154E 01	6.57 E 01	4.185E 00
9.207E 01	6.58 E 01	4.187E 00
9.260E 01	6.59 E 01	4.188E 00
9.319E 01	6.60 E 01	4.190E 00

FIG. 3 SAMPLE DATA

APPENDIX C

DETAILED OUTPUT

The output from the program is best described by reference to Figure 4, which is the output for the problem given in Appendix B.

The first item listed is the form of the equation that has been fitted. Each independent variable is listed as a component of each term but always with an exponent of zero wherever the variable does not appear explicitly in that term. The other items of the output are labeled and are self-explanatory. All items beyond the "Standard Error of Estimate" are printed at the option of the user.

```
                              SAMPLE PROBLEM

                            SELECTION NUMBER  1


( LN(Y)** 1.00) = B( 1)*(    (X(1))** 0.0 )*(    (X(2))** 0.0 )
                 +B( 2)*( LN(X(1))** 1.00)*(    (X(2))** 0.0 )
                 +B( 3)*(    (X(1))** 1.00)*(    (X(2))** 0.0 )
                 +B( 4)*(    (X(1))** 0.0 )*( LN(X(2))** 1.00)
```

FIG. 4 SAMPLE OUTPUT

```
                                  SAMPLE PROBLEM

                                SELECTION NUMBER  1

TERM      REGRESSION       STD. ERROR      COMPUTED        MEAN          STANDARD        CORRELATION
          COEFFICIENTS     OF REG. COEF.   T VALUE                       DEVIATION       F(X) VS H(Y)
C 1       -0.56414D 01     0.12375D 01     -4.55862
  2       -0.15476D 02     0.27009D 01     -5.72995     0.40352D 01    0.19732D 00        0.99588
  3        0.85649D-01     0.11781D-01      7.27021     0.57517D 02    0.96737D 01        0.99791
  4        0.48403D 02     0.82101D 01      5.89552     0.13938D 01    0.51228D-01        0.99318
H(Y)                                                    0.43015D 01    0.24872D 00

                         MULTIPLE CORRELATION     0.99946

                         STD.ERROR OF ESTIMATE    0.86282D-02


                     ANALYSIS OF VARIANCE FOR THE REGRESSION

     SOURCE OF VARIATION             DEGREES       SUM OF          MEAN            F VALUE
                                    OF FREEDOM     SQUARES        SQUARES

ATTRIBUTABLE TO REGRESSION              3        0.17920D 01    0.59735D 00     0.80240D 04
DEVIATION FROM REGRESSION              26        0.19356D-02    0.74445D-04
      TOTAL                            29        0.17940D 01


     CORRELATION COEFFICIENTS,MEANS,STANDARD DEVIATIONS,MAX VALUES,MIN VALUES,RANGES


                      X(1)              X(2)              Y(1)
           X(1)       1.00000
           X(2)       0.99325           1.00000
           Y(1)       0.99183           0.97313           1.00000
           MEAN       0.57517E 02       0.40352E 01       0.75801E 02
           STD DEV    0.96737E 01       0.19736E 00       0.15993E 02
           MAX        0.66000E 02       0.41900E 01       0.93190E 02
           MIN        0.30000E 02       0.34010E 01       0.34630E 02
           RANGE      0.36000E 02       0.78900E 00       0.58560E 02

                                 TABLE OF RATIOS

          Y VALUE     Y ESTIMATE     RATIO        X(1)          X(2)          WT


  1      3.5447E 00   3.5406D 00   1.0012E 00   3.0000D 01   3.4010D 00   1.0000D 00
  2      3.7201E 00   3.7268D 00   9.9822E-01   3.5000D 01   3.5550D 00   1.0000D 00
  3      3.8733E 00   3.8794D 00   9.9841E-01   4.0000D 01   3.6890D 00   1.0000D 00
  4      4.0086E 00   4.0089D 00   9.9992E-01   4.5000D 01   3.8070D 00   1.0000D 00
  5      4.1300E 00   4.1235D 00   1.0016E 00   5.0000D 01   3.9120D 00   1.0000D 00
  6      4.1536E 00   4.1496D 00   1.0010E 00   5.1000D 01   3.9320D 00   1.0000D 00
  7      4.1764E 00   4.1680D 00   1.0020E 00   5.2000D 01   3.9510D 00   1.0000D 00
  8      4.1992E 00   4.1911D 00   1.0019E 00   5.3000D 01   3.9700D 00   1.0000D 00
  9      4.2214E 00   4.2186D 00   1.0007E 00   5.4000D 01   3.9890D 00   1.0000D 00
 10      4.2435E 00   4.2382D 00   1.0013E 00   5.5000D 01   4.0070D 00   1.0000D 00
 11      4.2651E 00   4.2619D 00   1.0007E 00   5.6000D 01   4.0250D 00   1.0000D 00
 12      4.2870E 00   4.2896D 00   9.9939E-01   5.7000D 01   4.0430D 00   1.0000D 00
 13      4.3089E 00   4.3092D 00   9.9993E-01   5.8000D 01   4.0600D 00   1.0000D 00
 14      4.3306E 00   4.3326D 00   9.9954E-01   5.9000D 01   4.0770D 00   1.0000D 00
 15      4.3525E 00   4.3595D 00   9.9838E-01   6.0000D 01   4.0940D 00   1.0000D 00
 16      4.3749E 00   4.3900D 00   9.9657E-01   6.1000D 01   4.1110D 00   1.0000D 00
 17      4.3981E 00   4.4120D 00   9.9686E-01   6.2000D 01   4.1270D 00   1.0000D 00
 18      4.4229E 00   4.4373D 00   9.9676E-01   6.3000D 01   4.1430D 00   1.0000D 00
 19      4.4504E 00   4.4658D 00   9.9655E-01   6.4000D 01   4.1590D 00   1.0000D 00
 20      4.4847E 00   4.4858D 00   9.9976E-01   6.5000D 01   4.1740D 00   1.0000D 00
 21      4.4884E 00   4.4937D 00   9.9882E-01   6.5100D 01   4.1760D 00   1.0000D 00
 22      4.4920E 00   4.4901D 00   1.0004E 00   6.5200D 01   4.1770D 00   1.0000D 00
 23      4.4971E 00   4.4981D 00   9.9978E-01   6.5300D 01   4.1790D 00   1.0000D 00
 24      4.5017E 00   4.5062D 00   9.9900E-01   6.5400D 01   4.1810D 00   1.0000D 00
 25      4.5065E 00   4.5027D 00   1.0008E 00   6.5500D 01   4.1820D 00   1.0000D 00
 26      4.5115E 00   4.5108D 00   1.0002E 00   6.5600D 01   4.1840D 00   1.0000D 00
 27      4.5168E 00   4.5073D 00   1.0021E 00   6.5700D 01   4.1850D 00   1.0000D 00
 28      4.5225E 00   4.5155D 00   1.0016E 00   6.5800D 01   4.1870D 00   1.0000D 00
 29      4.5283E 00   4.5121D 00   1.0036E 00   6.5900D 01   4.1880D 00   1.0000D 00
 30      4.5346E 00   4.5203D 00   1.0032E 00   6.6000D 01   4.1900D 00   1.0000D 00
```

FIG. 4 Continued

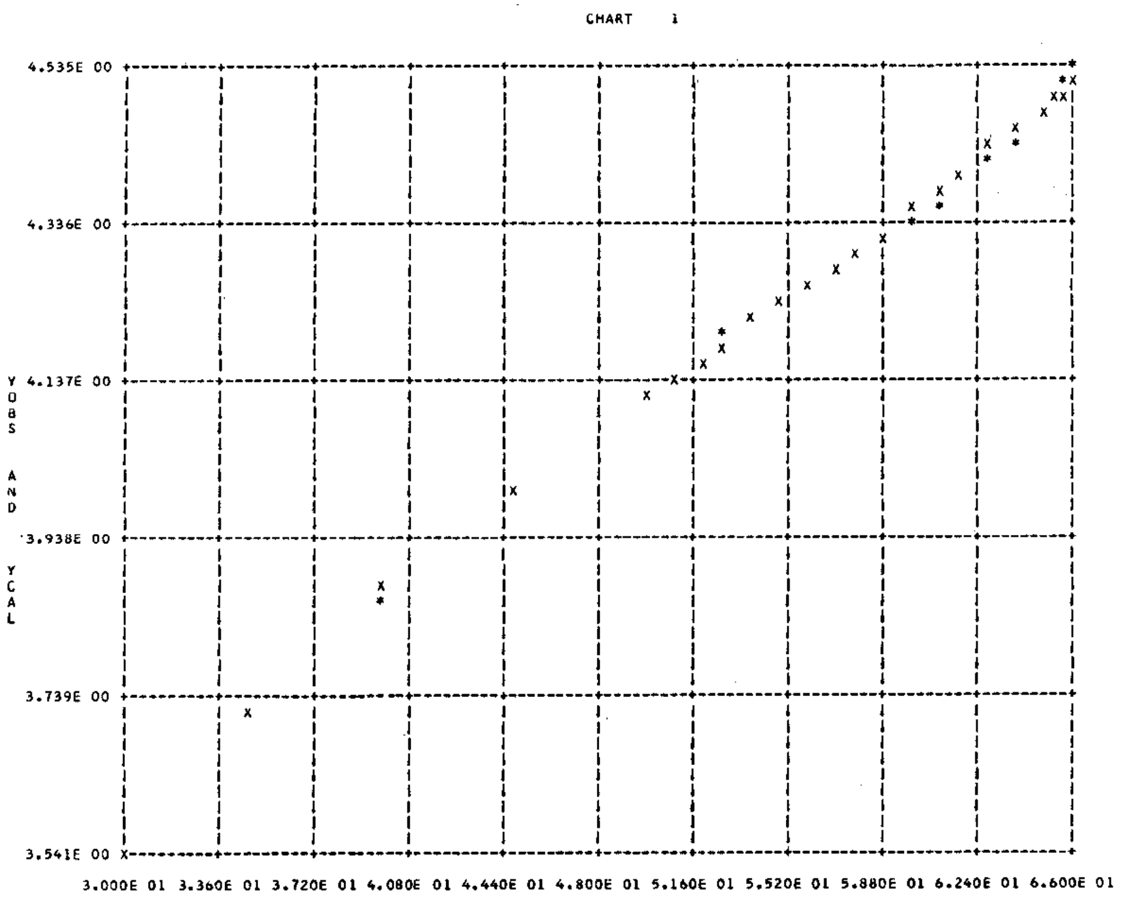


FIG. 4 Continued

```
                                   SAMPLE PROBLEM

                                 SELECTION NUMBER  2


 ( LN(Y)** 1.00) = B( 1)*(   (X(1))** 0.0 )*(   (X(2))** 0.0 )

                  +B( 3)*(   (X(1))** 1.00)*(   (X(2))** 0.0 )

                  +B( 4)*(   (X(1))** 0.0 )*( LN(X(2))** 1.00)


                                   SAMPLE PROBLEM

                                 SELECTION NUMBER  2

TERM    REGRESSION     STD. ERROR     COMPUTED      MEAN         STANDARD      CORRELATION
        COEFFICIENTS   OF REG. COEF.  T VALUE                    DEVIATION     F(X) VS H(Y)
C 1     0.13234D 01    0.23457D 00    5.64177
  3     0.18454D-01    0.16612D-02   11.10881    0.57517D 02   0.96737D 01     0.99791
  4     0.13751D 01    0.31370D 00    4.38358    0.13938D 01   0.51228D-01     0.99318
H(Y)                                             0.43015D 01   0.24872D 00

                          MULTIPLE CORRELATION    0.99878

                          STD.ERROR OF ESTIMATE   0.12736D-01


                                   TABLE OF RATIOS
                Y VALUE     Y ESTIMATE     RATIO         X(1)          X(2)          WT



  1          3.5447E 00   3.5603D 00   9.9563E-01   3.0000D 01   3.4010D 00   1.0000D 00
  2          3.7201E 00   3.7135D 00   1.0018E 00   3.5000D 01   3.5550D 00   1.0000D 00
  3          3.8733E 00   3.8566D 00   1.0043E 00   4.0000D 01   3.6890D 00   1.0000D 00
  4          4.0086E 00   3.9922D 00   1.0041E 00   4.5000D 01   3.8070D 00   1.0000D 00
  5          4.1300E 00   4.1219D 00   1.0020E 00   5.0000D 01   3.9120D 00   1.0000D 00
  6          4.1536E 00   4.1473D 00   1.0015E 00   5.1000D 01   3.9320D 00   1.0000D 00
  7          4.1764E 00   4.1724D 00   1.0010E 00   5.2000D 01   3.9510D 00   1.0000D 00
  8          4.1992E 00   4.1975D 00   1.0004E 00   5.3000D 01   3.9700D 00   1.0000D 00
  9          4.2214E 00   4.2225D 00   9.9975E-01   5.4000D 01   3.9890D 00   1.0000D 00
 10          4.2435E 00   4.2471D 00   9.9914E-01   5.5000D 01   4.0070D 00   1.0000D 00
 11          4.2651E 00   4.2717D 00   9.9844E-01   5.6000D 01   4.0250D 00   1.0000D 00
 12          4.2870E 00   4.2963D 00   9.9783E-01   5.7000D 01   4.0430D 00   1.0000D 00
 13          4.3089E 00   4.3206D 00   9.9731E-01   5.8000D 01   4.0600D 00   1.0000D 00
 14          4.3306E 00   4.3448D 00   9.9674E-01   5.9000D 01   4.0770D 00   1.0000D 00
 15          4.3525E 00   4.3689D 00   9.9623E-01   6.0000D 01   4.0940D 00   1.0000D 00
 16          4.3749E 00   4.3931D 00   9.9585E-01   6.1000D 01   4.1110D 00   1.0000D 00
 17          4.3981E 00   4.4169D 00   9.9576E-01   6.2000D 01   4.1270D 00   1.0000D 00
 18          4.4229E 00   4.4407D 00   9.9601E-01   6.3000D 01   4.1430D 00   1.0000D 00
 19          4.4504E 00   4.4644D 00   9.9686E-01   6.4000D 01   4.1590D 00   1.0000D 00
 20          4.4847E 00   4.4878D 00   9.9930E-01   6.5000D 01   4.1740D 00   1.0000D 00
 21          4.4884E 00   4.4903D 00   9.9957E-01   6.5100D 01   4.1760D 00   1.0000D 00
 22          4.4920E 00   4.4925D 00   9.9989E-01   6.5200D 01   4.1770D 00   1.0000D 00
 23          4.4971E 00   4.4950D 00   1.0005E 00   6.5300D 01   4.1790D 00   1.0000D 00
 24          4.5017E 00   4.4975D 00   1.0009E 00   6.5400D 01   4.1810D 00   1.0000D 00
 25          4.5065E 00   4.4997D 00   1.0015E 00   6.5500D 01   4.1820D 00   1.0000D 00
 26          4.5115E 00   4.5022D 00   1.0021E 00   6.5600D 01   4.1840D 00   1.0000D 00
 27          4.5168E 00   4.5044D 00   1.0028E 00   6.5700D 01   4.1850D 00   1.0000D 00
 28          4.5225E 00   4.5069D 00   1.0035E 00   6.5800D 01   4.1870D 00   1.0000D 00
 29          4.5283E 00   4.5090D 00   1.0043E 00   6.5900D 01   4.1880D 00   1.0000D 00
 30          4.5346E 00   4.5115D 00   1.0051E 00   6.6000D 01   4.1900D 00   1.0000D 00
```

FIG. 4 Continued

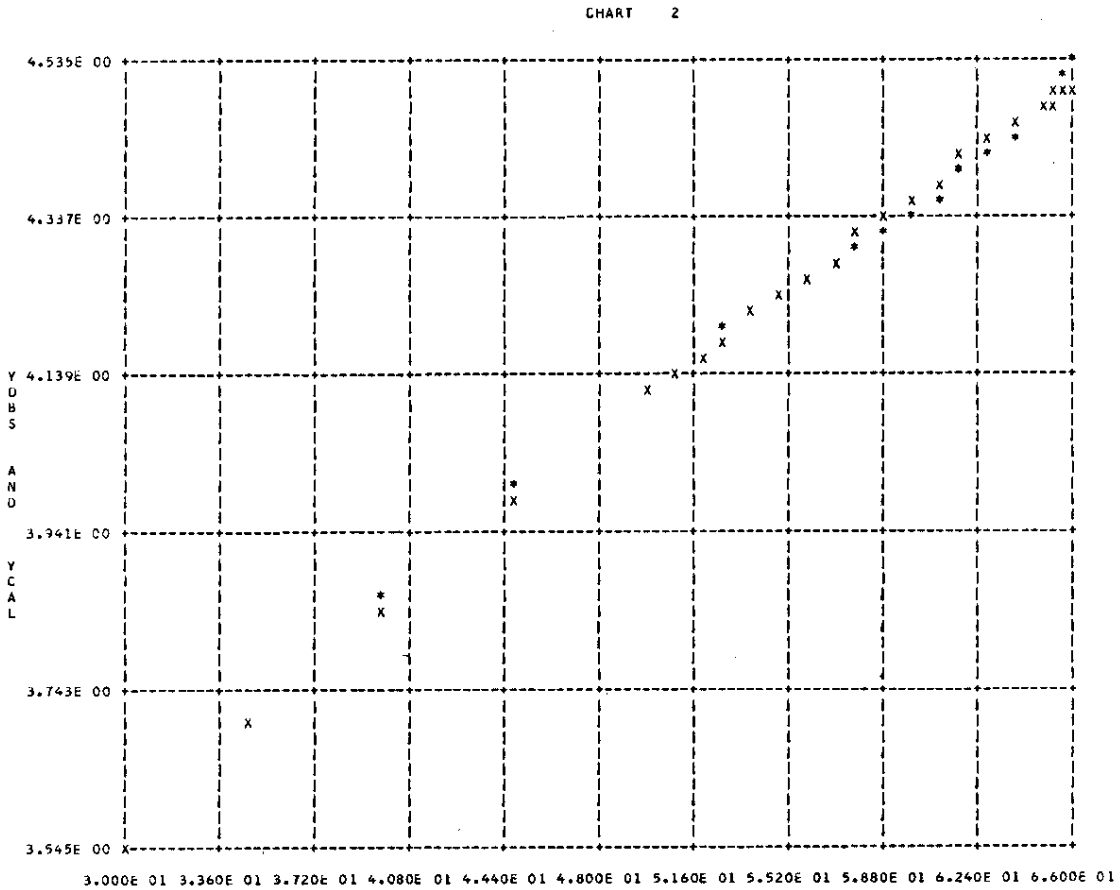


FIG. 4 Continued