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Higher Mode Frequency Effects on Resonance in Machinery, Structures, and Pipe Systems

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ABSTRACT

The complexities of resonance in multi-degree of freedom systems (multi-DOF) may be clarified using graphic presentations. Multi-DOF systems represent actual systems, such as beams or springs, where multiple, higher order, natural frequencies occur. Resonance occurs when a cyclic load is applied to a structure, and the frequency of the applied load equals one of the natural frequencies. Both equations and graphic presentations are available in the literature for single degree of freedom (SDOF) systems, which describe the response of spring-mass-damper systems to harmonically applied, or cyclic, loads. Loads may be forces, moments, or forced displacements applied to one end of a structure. Multi-DOF systems are typically described only by equations in the literature, and while equations certainly permit a case by case analysis for specific conditions, graphs provide an overall comprehension not gleaned from single equations. In fact, this collection of graphed equations provides novel results, which describe the interactions between multiple natural frequencies, as well as a comprehensive description of increased vibrations near resonance.

KEYWORDS

Resonance, multi degree of freedom, single degree of freedom, transmissibility, dynamic stress, bending of beams, critical speeds.

NOMENCLATURE

DOF	degree of freedom
E	modulus of elasticity, Newtons/ meter^2
F_0	force, Newtons
I	moment of inertia, meters^4
N_c	number of coils
St	static stress, Newtons / meters^2
TR	transmissibility
L	length, meters
c	damping coefficient, Newtons/ meter/second
$c_{critical}$	critical damping coefficient, Newtons/ meter/second
d	diameter, meters
f	forcing frequency, cycles / second
f_n	natural frequency, cycles / second
j	constant
k	spring constant, Newtons / meter
m	mass, kilogram
p	period, seconds
r	radius, meters
t	time, seconds
x	variable displacement of a mass, meters
x_0	constant vibration amplitude of a forcing function, meters
x_f	initial displacement from equilibrium, meters
x_{max}	maximum vibration amplitude for a mass, meters
y	variable displacement of a support, meters
y_0	constant vibration amplitude of a displacement function, meters
y_{max}	maximum vibration amplitude for a support, meters
z	distance, meters
ω	excitation force frequency, radian / second
ω_n	natural frequency , radian / second
ω_i	modal frequencies: $\omega_1, \omega_2, \ldots\omega_\infty$, radian / second
ϕ	phase angle for steady state vibration, radians
ρ	mass density, kg / meter^3
σ_{max}	maximum dynamic stress, Newtons / meters^2
θ	phase angle for free vibration, radians
ζ	damping factor

INTRODUCTION

To simplify this discussion of resonance, the maximum values of harmonic vibrations will be considered for linear, elastic structures. The structural material is assumed to be elastic and linear, where elastic materials will not permanently deform, and linear materials have a constant stiffness. When a structure vibrates due to a varying applied load, the structure vibrates at multiple frequencies. These natural frequencies are known for many simple structures, such as axially loaded rods and helical springs; and transversely loaded beams with different support conditions, such as fixed or free ends. The vibration at any point in a structure is simply the sum of the vibrations acting at that point due to each of the vibrations, or responses, of each modal frequency caused by the forcing function, or excitation force, which may be either a changing force applied to the structure, or a changing position, or displacement, of the structural supports. For the examples considered here, the applied forces and displacements are assumed to act as steady state sinusoidal functions.

When the frequency of the forcing function equals a natural frequency of a structure, resonance exists, and in the absence of damping the structure vibrates with an infinite magnitude. However, all real structures have some damping and the maximum vibration is limited. Even so, the response of the structure is magnified. For example, common helical springs frequently have transmissibilities (*TR*) exceeding 100 (Thomson [1]), where the transmissibility equals the transmitted force divided by the applied force. For the case of a spring with *TR* = 100, a weight slowly added to the end of a spring will stretch the spring to a specific length at rest, or static equilibrium. If that same weight is applied using a sinusoidal forcing function at resonance, the maximum length of the spring equals 100 times the length of the spring at rest.

Although this example is drastic, a typical implication is that acceptable vibrations may be greatly magnified and cause equipment damage if resonant conditions exist. Resonance will first be considered here for simple systems with a single natural frequency, followed by consideration of systems with multiple natural, or modal, frequencies, i.e., first mode, second mode, etc.

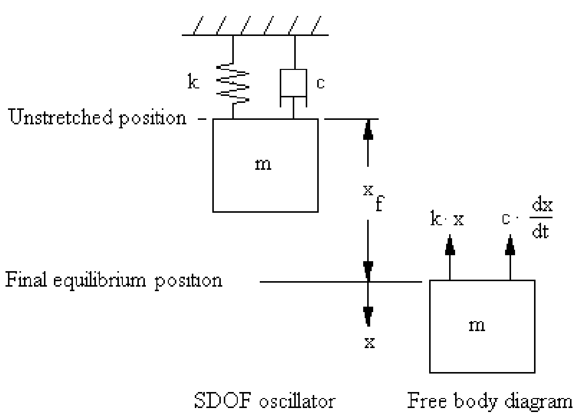

Figure 1: Free Vibration (Thomson [1])

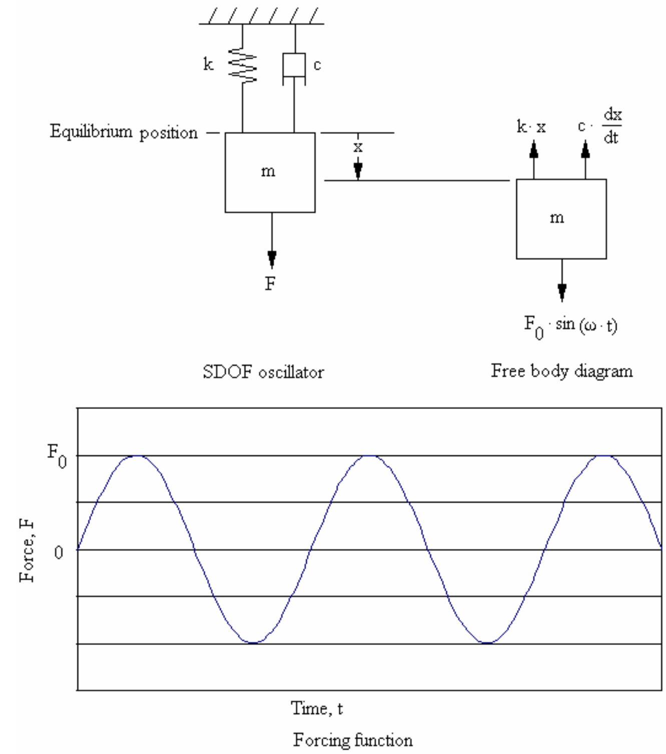

Figure 2: Load Controlled SDOF Oscillator
(Thomson [1])

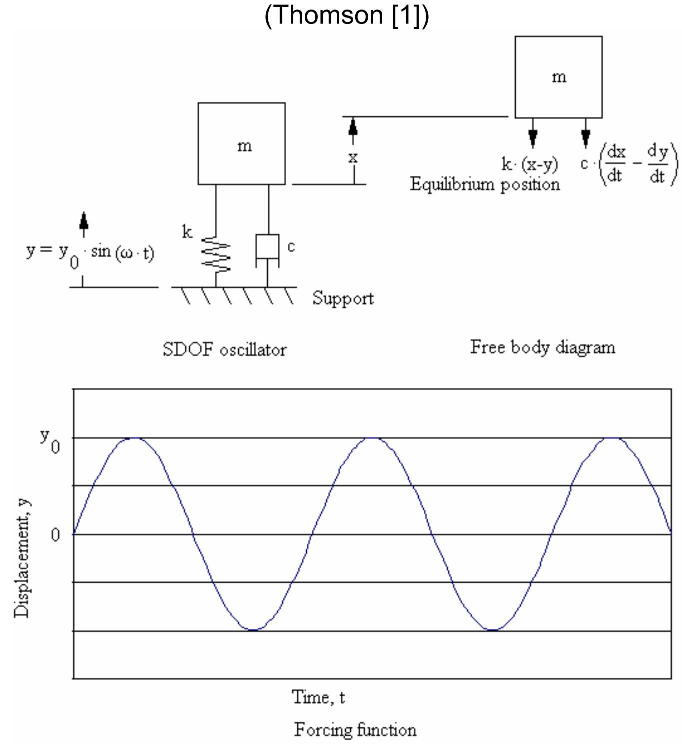

Figure 3: Displacement Controlled SDOF Oscillator
(Thomson [1])

SDOF SYSTEMS

To introduce resonance, single degree of freedom (SDOF) systems are used as examples. Even though these examples are presented in detail in the literature (Thomson [1]), they are summarized here to lead into a discussion of multi-DOF systems. Structures are simplified as SDOF oscillators, consisting of a spring, a mass, and a damper, as shown in Figs. 1 - 2. The oscillator is assumed to vibrate in response to an excitation force: which can be a changing force (load control), a changing displacement of the supports (displacement control), or a free vibration, which represents stretching the spring to its equilibrium position and releasing the mass.

SDOF Equation of Motion

The dynamic response of an oscillator is described in terms of Newton's equation of motion.

$$m \cdot \frac{d^2x}{dt^2} + c \cdot \frac{dx}{dt} + k \cdot x = f(t) \tag{1}$$

where *m* is the mass of the object; *k* is the spring constant, c is a constant, linear damping coefficient; *f(t)* describes the excitation force as a function of time, *t*; and

$$\frac{d^2x}{dt^2} \tag{2}$$

equals the instantaneous acceleration of the mass, and

$$\frac{dx}{dt} \tag{3}$$

equals the instantaneous velocity of the mass. For constant, linear damping coefficients, Eq. 1 yields

$$m \cdot \frac{d^2x}{dt^2} + 2 \cdot \zeta \cdot \omega_n \cdot \frac{dx}{dt} + \omega_n^2 \cdot x = f(t) \tag{4}$$

$$\zeta = \frac{c}{2 \cdot m \cdot \omega_n} = \frac{c}{c_{critical}} \tag{5}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 2 \cdot \pi \cdot f_n \tag{6}$$

$$p_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} \tag{7}$$

where ω_n is the natural frequency in radians / second; f_n is the natural frequency in cycles per second; p_n equals the period of the natural frequency which is the time between successive peaks of vibration; $c_{critical}$ is the critical damping factor; and ζ is the damping factor. Damping and frequency effects can be described using a free vibration equation.

SDOF Free Vibration

To evaluate free vibration, assume that the spring has an initial displacement, x_f, such that $x = x_f$ = constant at t = 0. Equation 4 yields

$$m \cdot \frac{d^2x}{dt^2} + 2 \cdot \zeta \cdot \omega_n \cdot \frac{dx}{dt} + \omega_n^2 \cdot x = 0 \tag{8}$$

$$x = x_f \cdot e^{-\zeta \cdot \omega \cdot t} \cdot \sin\left(\sqrt{1-\zeta^2} \cdot \omega_n \cdot t + \phi\right) \tag{9}$$

$$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2} \tag{10}$$

$$p_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} \tag{11}$$

where ω_d, f_d, p_d are the circular frequency, frequency, and period for damped vibration, respectively; and ϕ is the phase angle.

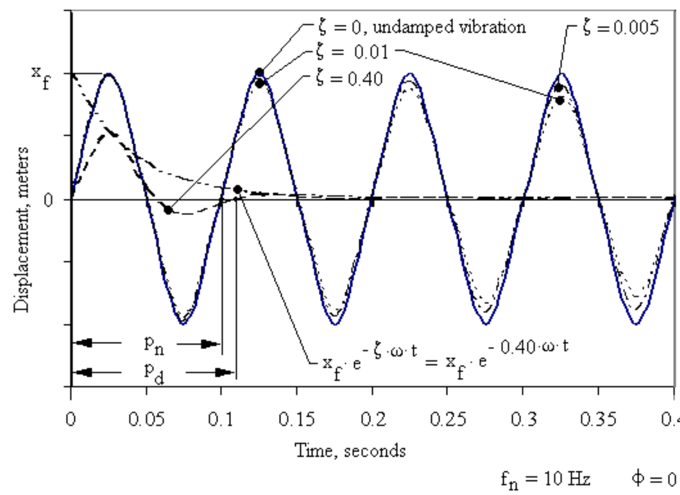

Figure 4: Example of Damped and Undamped Free Vibrations, Eq. 9

Damping effects. Free vibrations are graphed in Fig. 4 to demonstrate the effects of damping and clarify nomenclature. In the figure, p_d is shown for 40% damping, since p_d for 1% damping is near the undamped period, p_n, of the natural frequency. Different damping values are presented, where 0.5 - 6.0% damping is the range of damping values for steel structures, 7 – 145 % is the range for concrete structures, 15 - 40 % is the range for masonry structures (Pilkey [2]), and damping for helical springs drops below 0.005% (Harris and Piersol [3]). Note that $e^{-\zeta \cdot \omega_n \cdot t}$ bounds each sinusoidal response, and varies with the damping ratio, ζ. The exponential function is only shown for the 40% damping case.

Damping ratio. Vibrations in structures occur for positive damping ratios, $1 > \zeta > 0$. Vibrations are critically damped when $c = c_{critical}$, $\zeta = 1$, and vibrations do not occur. Below $\zeta = 1$, nonoscillatory motion occurs and the system returns to equilibrium. Between $\zeta = 0$ and $\zeta = 1$, vibrations decay to equilibrium in the absence of a forcing function. For $\zeta = 0$, the system is undamped and constant amplitude, natural frequency vibrations theoretically continue unabated even in the absence of a forcing function. For $\zeta < 0$, the system is unstable, and once initiated, vibrations increase without limit in the absence of a forcing function. Although numerous damping models are available, constant damping used here provides many simplifications to approximate structural behavior.

Phase angle effects. The relationship of phase angle to vibration is shown in Fig. 5. As a special case, a spring is stretched to a length, x_f, and then released. Then $\phi = \pi / 2$, and the response is shown in the figure. An arbitrarily selected phase angle of $\phi = \pi / 4$ is also shown in the figure.

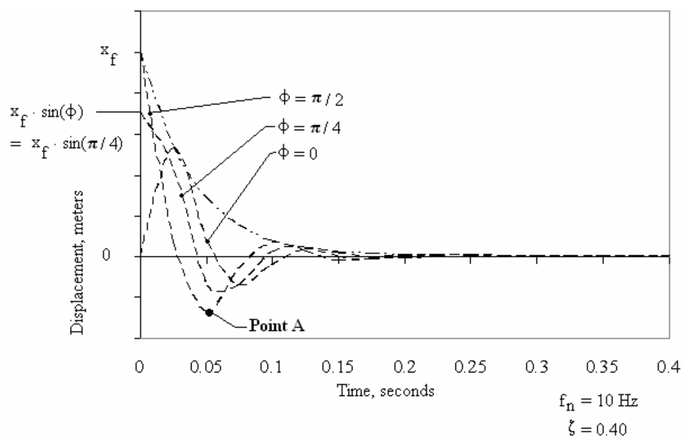

Figure 5: Phase Angle Effects on Vibration Amplitude, Eq. 9

Having defined nomenclature, Eq. 4 provides the basis for load or displacement controlled equations of motion. The above mathematical relationships for periods, frequencies, and damping ratios are, of course, the same for all vibrations considered here.

SDOF Load Control

Load control is shown in Fig. 1, and to describe the oscillator motion assume that

$$m \cdot \frac{d^2x}{dt^2} + 2 \cdot \zeta \cdot \omega_n \cdot \frac{dx}{dt} + \omega_n^2 \cdot x = F_0 \cdot \sin(\omega \cdot t) \qquad (12)$$

To solve this differential equation, a particular solution for the displacement is assumed to be

$$x = x_0 \cdot \sin(\omega \cdot t - \theta) \qquad (13)$$

where ω is the circular frequency of the applied force, x_0 is the amplitude of the impressed vibration, and θ is the phase of the displacement vibration with respect to the phase of the excitation force. Note that θ and ϕ may be the same, but need not be equal, depending on the boundary conditions. Equations 12 and 13 can be solved to yield the displacement at any time, t, to yield

$$x = \frac{F_0 \cdot \sin(\omega \cdot t - \theta)}{k \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}} + x_f \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin\left(\sqrt{1 - \zeta^2} \cdot \omega_n \cdot t + \phi\right) \qquad (14)$$

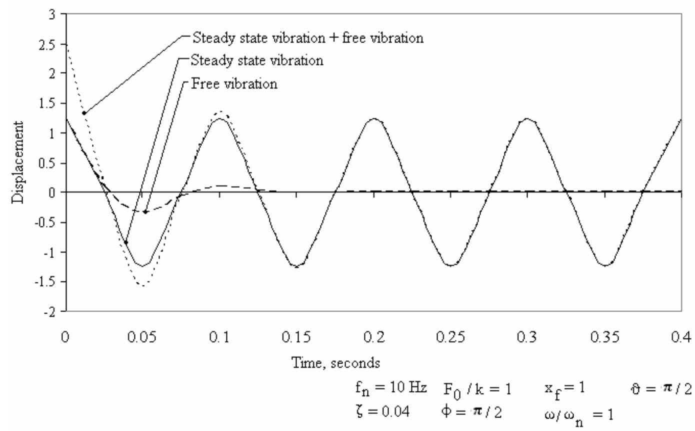

Figure 6: SDOF Load Controlled Vibrations, Free Vibration and Steady State Vibration, Eq. 14

An example of load control (Eq. 14) is shown in Fig. 6. This example displays the vibration when a spring is stretched, and the applied load is in phase with the free vibration, where the total initial force is applied at time, $t = 0$. Terms were arbitrarily selected: where the force / displacement ratio is expressed as a unit displacement ($x_0 = F_0 / k = 1$), the initial position is expressed as a unit displacement, $x_f = 1$, and the frequency was arbitrarily selected.

Compare the steady state vibration to the cumulative vibration shown in the figure to justify neglecting the transient free vibration. Note that the free vibration (for $\phi = \pi / 2$) has a decreasing effect on vibrations, and the transient vibration reduces x_{max}, which is the length that the spring is initially stretched. That is, the transient and steady state vibrations are additive from Eq. 14, and the negative transient vibration subtracts from the amplitude. In successive periods, the transient approaches zero. The damped transient affects vibration for only the initial vibration cycles, and for this example the damped vibration serves to increase the maximum amplitude. If θ and ϕ are equal and the vibrations are in phase, the free and steady state vibrations add, and the total vibration is higher during the initial cycles. In other words, the free vibration is typically neglected, but free vibration has an effect on vibration during the initial vibration cycles. For many cases, x_f equals zero and so does the free vibration.

Steady state, load controlled vibration. Considering only the steady state, forced vibration, the transient free vibration term may be neglected. To do so, assume that the system is initially at rest, and the spring is stretched to the equilibrium position, where $t = 0$, $x = x_f = 0$, and $dx/dt = 0$. Load controlled vibrations are then described, using

$$x = \frac{F_0 \cdot \sin(\omega \cdot t - \theta)}{k \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}} \qquad (15)$$

$$\phi = \tan^{-1}\left(\frac{2 \cdot \zeta \cdot \omega_n}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right) \qquad (16)$$

From Eq. 15, the maximum amplitude at resonance equals

$$x_{max} = \frac{F_0}{2 \cdot \zeta \cdot k} \qquad (17)$$

Figure 7 provides examples for the effects of forcing frequency. Vibrations with damping values of 1% are displayed for values of $\omega / \omega_n = 0.5$ and $\omega / \omega_n \approx 1.0$. A unit displacement, $1 = k / F_0$, is used to simplify Fig. 7, although unit displacement is atypical for most systems. Effects on both amplitude and phase are readily observed, since only the frequency of the impressed force is altered for the figure.

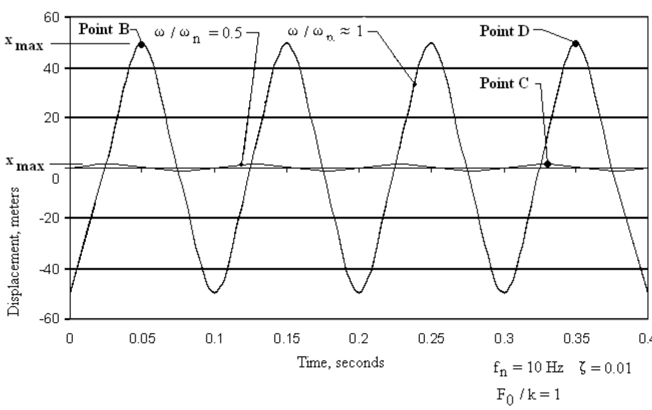

Figure 7: Example of Steady State, Load Controlled Vibrations, Eq. 15

Damping and frequency effects on transmissibility during load controlled vibration. Since a unit displacement was used in Fig. 7, transmissibility simply equals $TR = x_{max} = k \cdot x_{max} / F_0$, where transmissibility is defined as the maximum vibration amplitude divided by the static impressed force. When $\omega / \omega_n = 0$, the applied force is constant, the load is static, and $TR = 1$. For the examples in Fig. 6, the amplitude, x_{max} increases from $TR = 1.3$ at point C to $TR = 50$ at point D for 1% damping when ω / ω_n is doubled from 0.5 to 1.0. This large increase in amplitude clearly shows the effects at resonance.

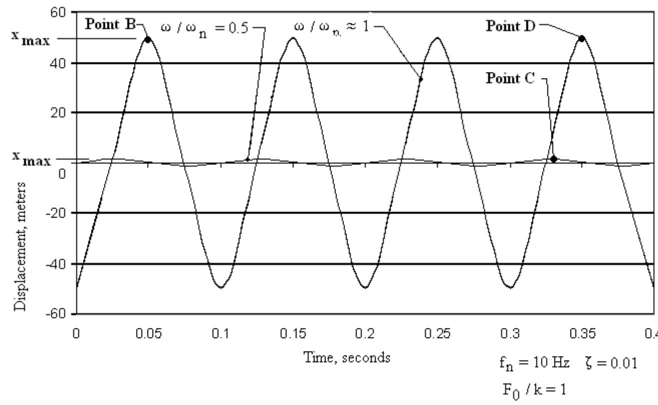

Figure 7: Example of Steady State, Load Controlled Vibrations, Eq. 15

Transmissibility for load control. To visualize transmissibility over a range of frequencies, non-dimensional transmissibility is typically presented in the literature. An expression for transmissibility is derived from Eq. 15, such that

$$TR = \frac{x_{max} \cdot k}{F_0} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}} \qquad (18)$$

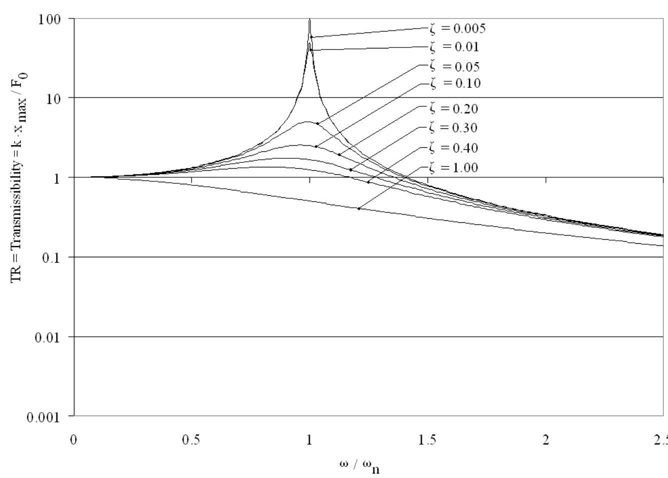

Figure 8: Transmissibility for Steady State Load Control, (Log Scale), Eq. 18 (Thomson [1])

Figure 8 logarithmically shows the effects of damping and transmissibilities which approach zero with increasing non-dimensional frequencies, ω / ω_n. Figure 9 shows the same data to scale.

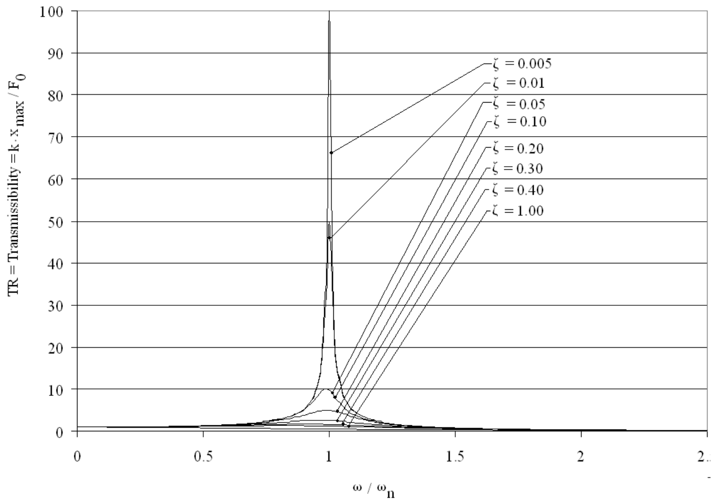

Figure 9: Transmissibility for Steady State Load Control, (Linear Scale), Eq. 18 (Thomson [1])

SDOF Displacement Control, Support Motion

For displacement control, consider Fig. 3. Assuming, $y = y_0 \cdot \sin(\omega \cdot t)$, the equation of motion is expressed as

$$m \cdot \frac{d^2(x-y)}{dt^2} + 2 \cdot \zeta \cdot \omega_n \cdot \frac{d(x-y)}{dt} + \omega_n^2 \cdot (x-y) = m \cdot \frac{d^2\left(y_0 \cdot \sin(\omega \cdot t)\right)}{dt^2} = m \cdot \omega^2 \cdot y_0 \cdot \sin(\omega \cdot t) \tag{19}$$

where y_0 is the vibration amplitude of the support, and x and y are the local coordinates for the mass and the support respectively. Neglecting free vibration,

$$x - y = \frac{m \cdot \omega^2 \cdot y_0 \cdot \sin(\omega \cdot t - \phi)}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}} \tag{20}$$

Then the steady state transmissibility is shown in Figure 10 and is expressed by Eq. 21, which closes this discussion on SDOF systems. There are other SDOF systems described in the literature, but consideration is limited here to load and displacement controlled vibrations as they are related to higher order vibrations.

$$TR = \left|\frac{x_{max}}{y_0}\right| = \sqrt{\frac{1 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_n}\right)^2}} \tag{21}$$

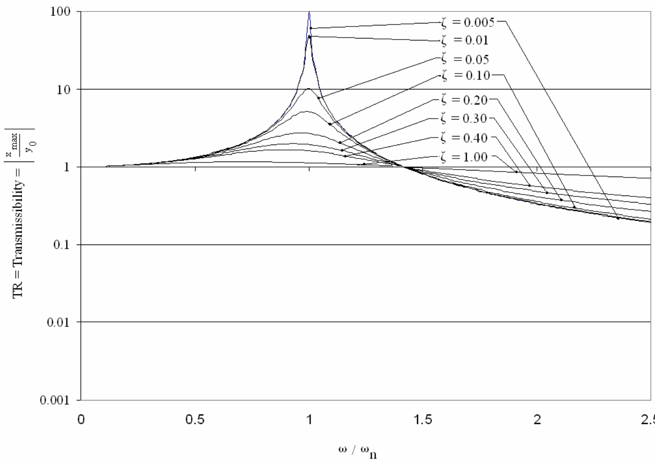

Figure 10: Transmissibility for Displacement Controlled Vibration (Support Motion), Eq. 21 (Thomson [1])

VIBRATIONS AND HIGHER ORDER FREQUENCIES

Structures are multi-DOF systems, where the degrees of freedom are a series of natural frequencies, $\omega_i = \omega_1, \omega_2, \ldots\omega_\infty$, at which a structure vibrates. For the case of linear damping, the equations of motion are uncoupled. That is, each higher mode frequency of a system may be considered independently. For this discussion, equations to describe higher mode vibration equations are developed here for both load and displacement controlled vibrations, and these equations are combined with available frequency equations for some structures to investigate modal effects on structural response.

Multi-DOF Load Control

For load controlled vibrations, some cases may be described by assuming that each frequency is described by a sin response, where

$$m \cdot \frac{d^2 x_i}{dt^2} + 2 \cdot \zeta \cdot \omega \cdot \frac{dx_i}{dt} + \omega \cdot x_i = F_0 \cdot \sin(\omega \cdot t) \tag{22}$$

$$x_i = \frac{F_0 \cdot \sin(\omega \cdot t - \theta)}{k_i \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_i}\right)^2}} \tag{23}$$

$$\omega_i^2 = \frac{k_i}{m} \Rightarrow k_i = \frac{\omega_i^2 \cdot k_1}{\omega_1^2} \tag{24}$$

where x_i and k_i are displacements and frequencies associated with each mode, i. These equations are used here to describe the maximum response of springs, axially loaded rods, and simply supported beams.

As shown in Fig. 11, the discrete modal vibration magnitudes add to yield the total vibration. Modal participation factors approximate the contribution of each mode to the vibration, but participation factors are available for only a few cases in the literature.

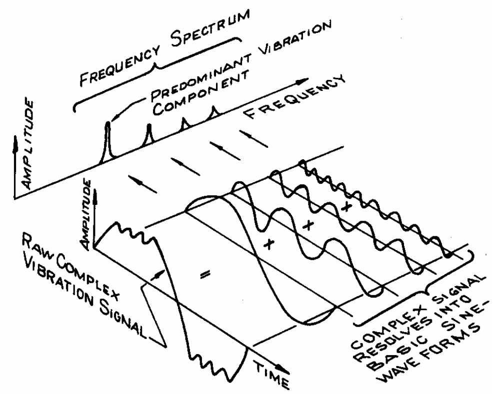

Figure 11: Vibration Frequencies (Karrassik [4])

Load Controlled Vibrations for Springs

Spring displacements can be described in terms of higher mode frequencies, where the frequencies for common steel springs are expressed as

$$\omega_i = \frac{i \cdot 22.05 \cdot d}{r^2 \cdot N_c} \tag{25}$$

where d is the diameter of the spring wire, r is the mean radius of the helix, and N_c is the number of active coils

Modal Contributions for Spring Vibrations. Figure 12 shows the contribution of other vibration frequencies to a fundamental mode, in this case the first mode. This contribution to the overall vibration is sometimes referred to as a participation factor, where each of the modal frequencies is affected by vibrations from other modes.

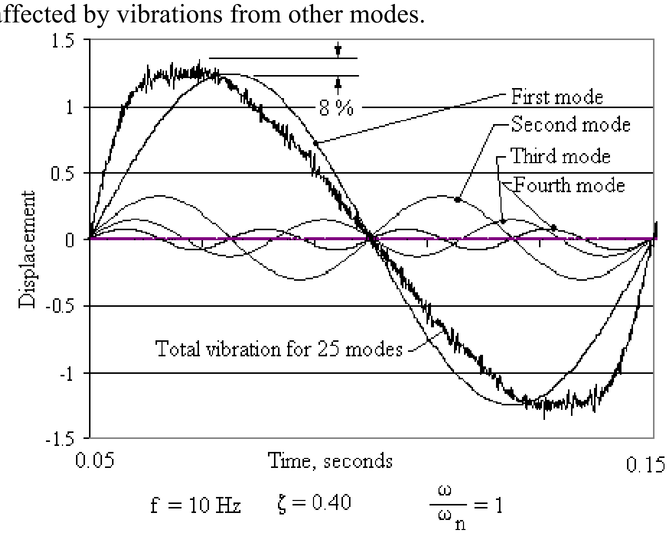

Figure 12: Participation of Higher Mode Frequencies for the First Mode for Springs, Eq. 28

To consider these effects due to other modes on a primary mode of vibration for springs, first substitute Eq. 24 into Eq. 23 to yield

$$x_i = \frac{\omega_1^2 \cdot F_0 \cdot \sin(\omega \cdot t - \theta)}{\omega_i^2 \cdot k_1 \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_i}\right)^2}} \tag{26}$$

Set $F_0 / k_1 = x_0 = 1$, and from Eq. 25 for a spring note that

$$\frac{\omega_1^2}{\omega_i^2} = \frac{1}{i^2} \tag{27}$$

Then, vibration is described for each mode as

$$x = \sum_{i=1}^{\infty} x_i \approx \sum_{i=1}^{j} x_i = \sum_{i=1}^{j} \frac{\sin(\omega \cdot t - \theta)}{i^2 \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_i}\right)^2}} \tag{28}$$

where j is selected to provide reasonable accuracy. In the example of Fig. 12, j equals 25. In this figure, the first four vibration modes are shown, along with the total vibration for the first 25 modes of vibration.

Resonance occurs for each mode, and each higher mode is treated similarly when the system is excited at that modal frequency. For example, when the second mode is excited, ω_2 is treated as the fundamental natural frequency, and ω_1 is half of ω_2, such that

$$\frac{\omega_1^2}{\omega_2^2} = \frac{1}{2^2} \tag{29}$$

and

$$x = \sum_{i=1}^{\infty} x_i \approx \sum_{i=1}^{j} x_i = \frac{\sin(\omega \cdot t - \theta)}{2^2 \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_i}\right)^2}} + \sum_{i=2}^{j} \frac{\sin(\omega \cdot t - \theta)}{i^2 \cdot \sqrt{\left(1 - \left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot \omega}{\omega_i}\right)^2}} \tag{30}$$

Modal participation factors for the second mode are comparable to the first mode factors shown in Fig. 12. The approximate 8 % contribution of higher modes to the total vibration shown in the figure is typical for springs. This higher mode effect can be calculated for each point on a transmissibility curve, but for the purposes of this paper the 8 % value is neglected when considering transmissibility.

Transmissibility Calculations for Vibrations of Springs. The example shown in Figs. 13 and 14 highlights a method to calculate trasnmissibilties due to higher mode frequencies. To explain the figures, each of the resonant frequencies is first plotted in Fig. 13, where Eq.18 is rewritten as

$$TR = \frac{x_{i,max} \cdot k_i}{F_0} = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2\cdot\zeta\cdot\omega}{\omega_i}\right)^2}} \qquad (31)$$

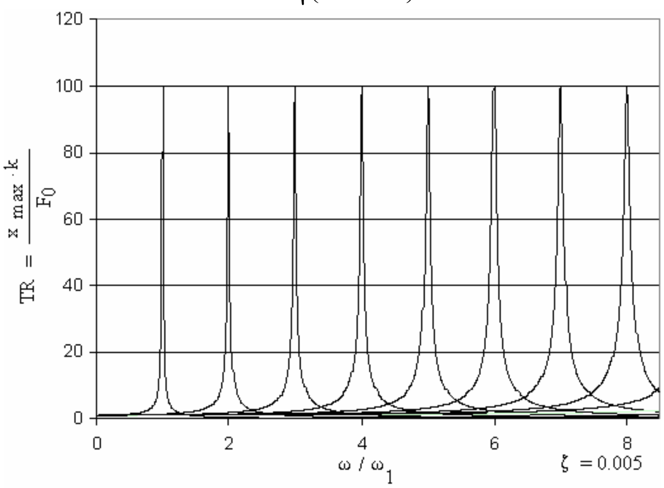

Figure 13: Example of Modal Frequencies for Load Control of a Spring, Eq. 31

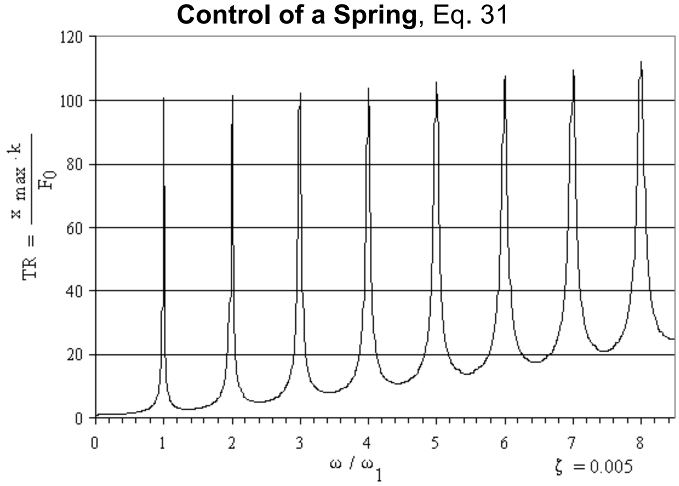

Figure 14: Transmissibility Example for Load Control of a Spring, Eqs. 32 and 33

Then, to relate transmissibilities for multiple modes, the vibration needs to be reconsidered. Each of the resonant vibrations acts separately, and the vibrations are therefore additive. However, Fig. 13 shows only the maximum transmissibility for each modal frequency. To add the frequencies, the fact must be recognized that the transmissibilities are referenced to static equilibrium, where *TR* = 1, and only the transmissibility magnitudes with respect to *TR* = 1are additive. Then the transmissibilities (Eq. 31) may be related by

$$\sum_{i=1}^{75} \frac{x_{i,max} \cdot k_i}{F_0} \approx TR = 1 + \sum_{i=1}^{\infty} \left(\left(\frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_i}\right)^2\right)^2 + \left(\frac{2\cdot\zeta\cdot\omega}{\omega_i}\right)^2}} \right) - 1 \right) \qquad (32)$$

For this example, the first 75 modes were used in the calculation to ensure reasonable accuracy for highly damped vibrations (ζ = 0.40), although only 8 modes are displayed in the figures. Then,

$$\sum_{i=1}^{75} \frac{x_{i,max} \cdot k_i}{F_0} \approx TR = 1 + \sum_{i=1}^{\infty} \left(\left(\frac{1}{\sqrt{\left(1-\left(\frac{1}{i}\right)^2\right)^2 + \left(\frac{2\cdot\zeta\cdot 1}{i}\right)^2}} \right) - 1 \right) \qquad (33)$$

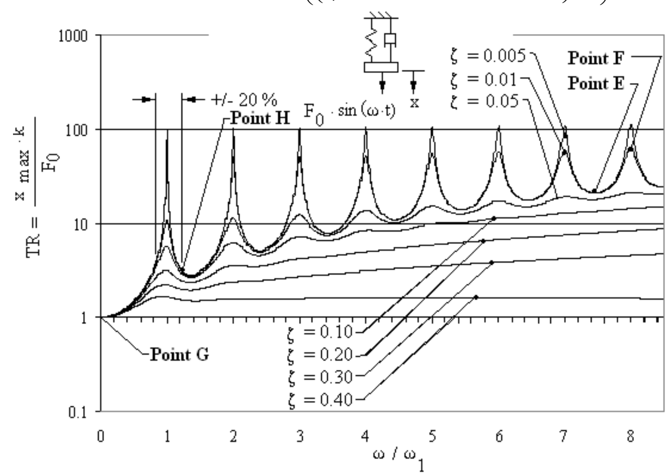

Figure 15: Multi-DOF Transmissibility for Load Control of Springs, Eqs. 32 and 33

Total Transmissibility for Vibrations of Springs. Transmissibilties are shown for different values of damping in Fig. 15. Substitution of different damping values into Eq. 33 yielded the figure. Comparing Fig. 8 to Fig. 15, the results are markedly different following the first mode vibration, with respect to increasing ω.

For example, resonance in machinery occurs at the critical speed, which is the speed, where the motor speed = $\omega = \omega_n$. Common misconceptions are that operating between resonant frequencies always corrects vibration problems, and that operating at least 10 – 20 % away from resonance always corrects vibration problems. Assume that a motor applies a force to a spring and consider point E, where the transmissibility equals 21. If equipment is operated under these conditions at ζ = 1 % between the seventh and eighth mode frequency, the best that can be done to minimize vibrations without added damping is to operate at point E. Then TR is reduced from 62 for resonance at point F to 21 at point E. In other words, any equipment vibration is still significantly amplified for lightly damped structures, such as machinery, piping, or steel buildings. Vibrations can be reduced by operating at point E, but vibration magnitudes are still significant. The problem may, or may not, be corrected by operating between frequencies. Point H presents the first mode frequency and operating at least 20 % away from resonance. The *TR* = 3.4, which again may, or may not be acceptable, depending on design and reliability requirements for the system of concern where point G represents static design.

Load Controlled Vibrations for Rods. Vibrations are similar for axially loaded rods which are fixed at one end and free to move at the other end. To calculate *TR*, Eq. 32 was used except that

$$\omega_i = \frac{(2\cdot i-1)\cdot\pi}{2\cdot L}\cdot\sqrt{\frac{E}{\rho}} \tag{34}$$

$$\sum_{i=1}^{75}\frac{x_{i,max}\cdot k_i}{F_0}\approx TR = 1+\sum_{i=1}^{\infty}\left(\left(\frac{1}{\sqrt{\left(1-\left(\frac{1}{(2\cdot i)-1}\right)^2\right)^2+\left(\frac{2\cdot\zeta\cdot 1}{(2\cdot i)-1}\right)^2}}\right)-1\right) \tag{35}$$

where *L* is the length of the rod, *E* is the modulus of elasticity, *ρ* is the mass density, and 75 modes were used to ensure calculation accuracy.

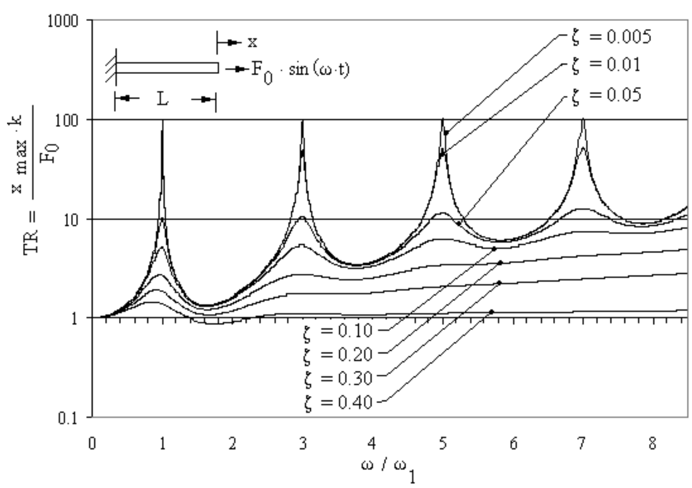

Figure 16: Transmissibility for Axially Loaded Rods, Eqs. 33 and 34

Figure 16 shows transmissibility for axially loaded rods, and Fig. 17 shows the effects of additional higher mode frequencies (Eq. 28), which are neglected here. However, note that the overall vibrations are reduced, while the vibrations were increased for a spring.

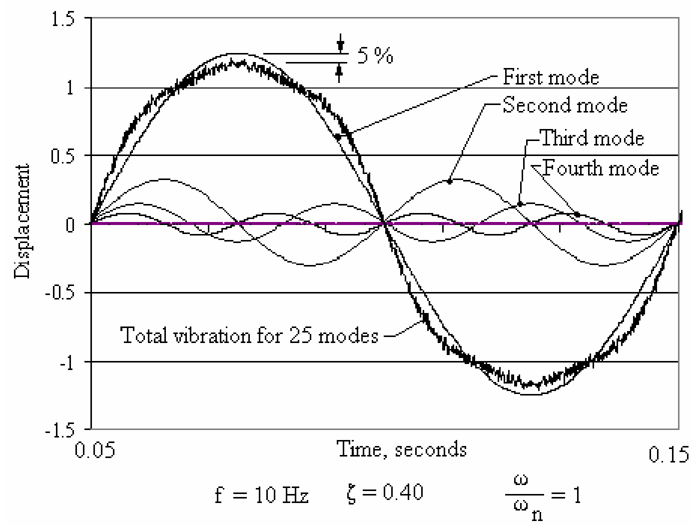

Figure 17: Participation of Higher Mode Frequencies for First Mode Response of a Loaded Rod, Eq. 28

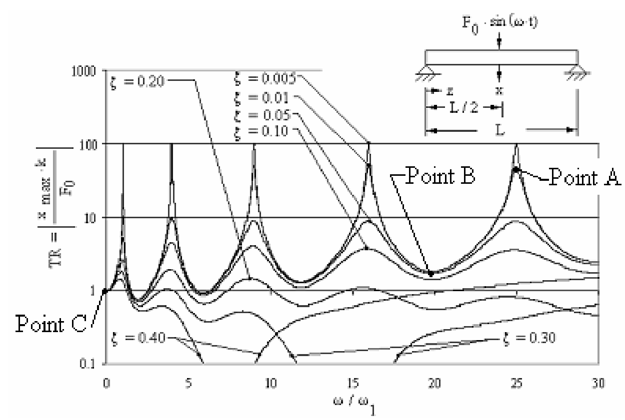

Figure 18: Transmissibility for a Simply Supported Beam, Eq. 38

Load Controlled Vibrations for Transversely Loaded Uniform Beams. The response for beams with different types of end supports can be evaluated using the techniques presented here, but this discussion is limited to the case of a beam with pinned, or hinged, supports. For this case, the mode shape and frequencies are listed by (Pilkey [2]), such that

$$\text{beam\_deflection\_mode\_shape} = \sin\left(\frac{i\cdot\pi\cdot z}{L}\right) \tag{36}$$

$$\omega_i = \frac{(i\cdot\pi)^2}{L^2}\cdot\sqrt{\frac{E\cdot I}{\rho}} \tag{37}$$

$$\sum_{i=1}^{75} \frac{x_{i,max} \cdot k_i}{F_0} \approx TR = \left| 1 + \sum_{i=1}^{\infty} \left(\left(\frac{1}{\sqrt{\left(1 - \left(\frac{1}{i^2}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot 1}{i^2}\right)^2}} \right) - 1 \right) \right| \quad (38)$$

where *I* is the moment of inertia, *L* is the length of the beam, and *z* is the distance along the beam from a support. The absolute value was used, since vibrations may be negative with respect to static equilibrium, but structural vibration motion is always a positive quantity.

To evaluate beam deflections with respect to resonance, assume that the vibration is sinusoidal, and that the maximum deflection occurs at the center of the beam. Both of these assumptions are consistent with Eq. 36. Then the equations presented for a spring are applicable, and all that is required is to substitute the appropriate frequencies into Eq. 32 to obtain Fig. 18. Note that as damping increases, the effects of resonance diminish significantly, due to structural stiffness.

Multi-DOF Displacement Control

An example of displacement control is provided in Fig. 19. Derivations comparable to those above are performed to describe support motion with respect to multi-DOF resonance, where

$$m \cdot \frac{d^2(x_i - y_i)}{dt^2} + 2 \cdot \zeta \cdot \omega_i \cdot \frac{d(x_i - y_i)}{dt} + \omega_i \cdot (x_i - y_i) = \quad (39)$$

$$m \cdot \omega^2 \cdot y_0 \cdot \sin(\omega \cdot t)$$

$$\sum_{i=1}^{75} \frac{x_{i,max} \cdot k_i}{F_0} \approx TR = 1 + \sum_{i=1}^{\infty} \left(\left(\sqrt{\frac{1 + \left(\frac{2 \cdot \zeta \cdot 1}{i}\right)^2}{\left(1 - \left(\frac{1}{i}\right)^2\right)^2 + \left(\frac{2 \cdot \zeta \cdot 1}{i}\right)^2}} \right) - 1 \right) \quad (40)$$

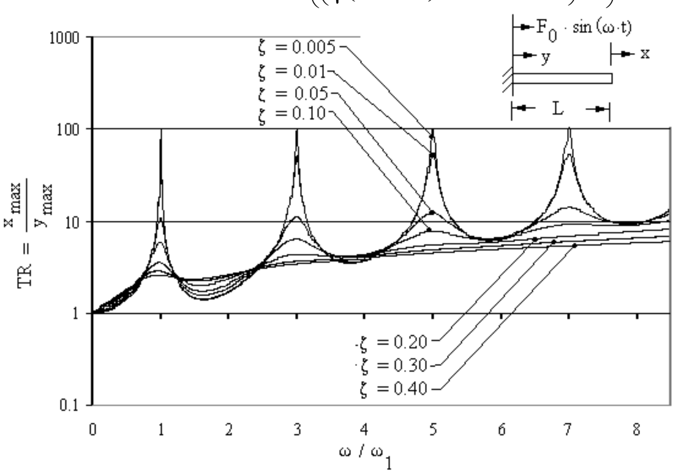

Figure 19: Support Motion for an Axially Loaded Rod, Eq. 40

Compare support motion (Fig. 19) and load control (Fig. 16) to note the similarities for light damping and differences for higher damping.

APPLICATION OF RESULTS

This new theory expands current, published text book vibration theory to address a specific area of vibration analysis, and the theory has direct applications to piping design. Although a specific piping example is not presented here, the basic theory is applicable to piping design, critical speed machinery operation, and numerous other applications.

For example, in a pipe system, any higher mode pipe frequency can be excited by an attached pump or compressor, and this paper explains the relationship between forcing frequencies and modal frequencies. The curves provided demonstrate that if a system is operating between modes, significant vibration still occurs.

In particular, piping vibrates in response to resonant vibrations and / or pump pulsations. These vibrations typically occur at the pump motor frequency or the vane pass frequency. A tacit assumption of vibration analysis is that any higher mode frequency of a structure, or pipe system, may be excited by a forcing function (excitation frequency). Some analysts simply identify that a critical speed, or vane pass frequency, equals a pipe frequency. That is, some analysts implicitly use the theory presented in this paper. Others do not recognize a relationship between forcing frequency and higher mode structural frequencies at all.

In earlier work with vibration problems, this issue was addressed during analysis of pump vibrations (Leishear [5]). Specifically, in the early 1990's a problem of this nature was solved, where the system piping vibrated at one of its higher mode natural frequencies. Using vibration analyzers, one natural frequency of the piping attached to a pump was determined to be a multiple of the motor speed. The entire pipe system in a pump house connected to a pump resonated at a higher mode pipe frequency. Vibration dampers were temporarily installed on the pump motor, but this approach inadequately reduced vibration. Finally, the base plate was stiffened which reduced the vibration sufficiently to eliminate pump mechanical seal failure problems. The use of an expansion joint was also considered, but the cost of system shutdown was prohibitive.

At the time, vibration analysis was not well developed in this area. Using classic vibration theory, a single resonant frequency is often assumed for the system response. Using that assumption, one concludes that operating away from the forcing frequency will resolve the problem, but it does not in many cases. The vibration may be reduced, but not eliminated as shown in the graphs. The point is that one's ability to understand vibration of systems and machinery is enhanced by an implicit understanding of system response as provided in the curves presented in the paper. Prior to this paper, the curves were "intuitive", based on evaluations of thousands of vibration plots, over the years. Current text books are inadequate to explain this phenomenon, and a published theoretical explanation is presented here to aid other vibration analysts. Any time that a pipe system vibrates in response to pump or compressor vibration, any higher mode frequency can be excited as approximately described by the figures in this paper. If one does not fully understand the basic principles, successful application of those principles is limited.

As another example, the use of a reciprocating pump is also a common source of pipe system vibration problems, where the pulsations from this type of pump may cause significant system vibrations. Inspection of Figure 18 provided in this paper quickly concludes that changing the pump speed has limited effects on a simply supported pipe. A fixed end pipe yields similar graphic results. Since piping vibrations are frequently in the range of 1 - 2 %, points A, B, and C can be considered. Assume that the pipe is independently determined to vibrate at a resonant fourth mode frequency, using vibration analyzers. Then point A represents the operating condition at mode 4. By changing the pump speed (forcing frequency), vibrations may be reduced to the level at point C, which is an order of magnitude reduction in vibration amplitude. However, vibrations cannot be reduced to static load conditions (point C) by simple speed reductions. In other word, first cut, quick estimates of potential changes in vibration can be quickly discerned using the graphs. Although outside the scope of this paper, the effects of stiffening on system response can also be quickly approximated (ball parked) using the graph. An example of stiffening effects on system response are available (Leishear [6]).

CONCLUSIONS

The goal of this paper is to provide the basic theory for future use and improved understanding of vibration mechanics of structures. Understanding the basic physics of higher mode frequency effects will aid other vibration analysts. The graphs provide a pictorial presentation of the issue, and derivations to support the graphs. In short, this new theory advances the understanding of vibration analysis by providing simplified graphs, which can be used to understand the overall system dynamics associated with higher mode frequencies in structures.

Resonance was considered for different components, such as vibrating springs, rods, and simply supported beams to better understand the effects of higher mode frequencies. Transmissibility was defined as the maximum value of the magnification of deflection when compared to the deflection for statically loaded components. Transmissibility was used to explain resonance along with critical speeds, where transmissibilities varied with respect to damping and frequency.

For many applications, operating near or above the first mode frequency results in magnified displacements, and potential equipment damage. Operating midway between critical speeds decreases the vibration, where critical speeds are defined as a condition where a structural frequency is coincident to an attached motor, pump, or compressor, which is attached to a structure. However, significant vibration may still be present. For a few cases of high damping, resonance effects can be nearly eliminated. Classic single mode, single degree of freedom approximations were shown to be incorrect, and higher mode frequency graphs corrected this shortcoming of current vibration theory.

In short, the first comprehensive explanation of the physics associated with multi-DOF resonance in real structures is presented to better understand machinery, piping, vehicle, and building failures due to vibration. Novel graphic presentations provide a level of understanding for this common failure mechanism not previously available, and this advancement of vibration theory is widely applicable to safer and more reliable equipment designs and applications.

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